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MATHEMATICS

«THE LARGE SIEVE»

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1. From Viggo Brun's investigations we know that if from the numbers $1, 2, \dots X$ we delete k classes of residues to every prime modulus p_i under the condition

$$1 \leqslant p_i \leqslant \sqrt{X} \tag{1}$$

the number of remaining numbers will not exceed

$$c_1(k)\frac{X}{\ln^k X}$$

$c_1(k)$ being a constant depending on k only.

This method, however, is not applicable to the estimation of the number of remaining numbers, if we delete not a constant number k of classes of residues for every p_i, but $f(p_i)$ classes of residues $(\mathrm{mod}\, p_i)$ where $f(p_i)$ increases with p_i.

We shall here discuss this question with a view to give some further applications to the theory of primes. We shall call the described operation «the large sieve».

2. Theorem I. *Suppose that we are given Z different integers M_1, M_2, M_Z between 1 and X. Consider all primes p_i between 1 and $\sqrt{X}$*

$$1 \leqslant p_i \leqslant \sqrt{X}. \tag{1}$$

Let, further, a function $f(p)$ be given, which is positive for $p > 0$ and satisfies the condition $f(p) \leqslant p$. Denote the min $\frac{f(p_i)}{p_i}$, p_i running over (1), by $\tau_X > 0$. Then for every prime p_i from (1) between the numbers $M_i\,(i = 1, 2, \dots, Z)$ there are at least $p_i - f(p_i)$ distinct classes of residues $(\mathrm{mod}\, p_i)$ with a possible exception of not more than

$$Y \leqslant 20\pi \frac{X}{\tau_X^2 Z} \tag{2}$$

numbers p_i.

Example. Let $M_i\,(i = 1, 2, \dots, Z)$ be primes; $M_i = p_i$, $f(p) = p^{\frac{3}{4}}$, then $\tau_X = \frac{1}{X^{\frac{1}{8}}}$ and $Y = 80X^{\frac{1}{4}} \ln X$ for sufficiently large X.

Theorem II (a consequence of I). *If we delete from the numbers $1, 2, \ldots, X$ the $f(p_i)$ classes of residues to each of Y arbitrarily fixed moduli p_i under (1), the number of remaining numbers will not exceed*

$$Z \leqslant 20\pi \frac{X}{\tau_X^2 Y}\cdot$$

Proof. Consider the sum $S(\alpha)=\sum_{j=1}^{Z} e^{2\pi i\alpha M_j}$. Then we have

$$\int_0^1 |S(\alpha)|^2\,d\alpha = Z. \tag{3}$$

Let p be one of the p_i and $\delta=\frac{\tau_X}{20\pi X}$. We form the integral

$$I_p=\int_{-\delta}^{\delta}\sum_{y=0}^{p-1}\left|S\left(\frac{y}{p}+x\right)\right|^2 dx. \tag{4}$$

The integrand can be written in the form

$$\sum_{y=0}^{p-1}\sum_{j,j'}^{Z} e^{2\pi i\left(\frac{y}{p}+x\right)(M_j-M_{j'})}.$$

Changing the order of summation and effectuating it, we find it to be equal to

$$p\sum_{M_j-M_{j'}\equiv 0 \pmod p} e^{2\pi i x(M_j-M_{j'})}=T_p>0$$

where j and j' run so that $M_j-M_{j'}\equiv 0 \pmod p$. Let $\xi_1, \xi_2, \ldots, \xi_s$ be all distinct residues (mod p) of the numbers $M_1, \ldots, M_Z$, and let ξ_1 occur among these numbers a_1 times, ξ_2 occur a_2 times, $\ldots$ ξ_s occur a_s times, so that $a_1+a_2+\ldots+a_s=Z$. We have further

$$\left|e^{2\pi i x(M_j-M_{j'})}-1\right|\leqslant\left|e^{\frac{2\pi\tau_X}{20\pi}}-1\right|<\frac{e}{10}\tau_X$$

$|x|$ being less than δ, so that we can write

$$T_p>p(a_1^2+\ldots+a_s^2)\left(1-\frac{e}{10}\tau_X\right).$$

Next we apply Schwarz' inequality in the form

$$(a_1^2+a_2^2+\ldots+a_s^2)\geqslant\frac{(a_1+\ldots+a_s)^2}{s}=\frac{Z^2}{s}$$

and find

$$T_p>Z^2\frac{p}{s}\left(1-\frac{e}{10}\tau_X\right).$$

Substituting into (4), we get

$$I_p>2\delta Z^2\left(1-\frac{e}{10}\tau_X\right). \tag{5}$$

Now let us consider the numbers p_i, for which there are less than $p_i-f(p_i)$ distinct residues (mod p_i) between the numbers M_j; for such a number p we have

$$s\leqslant p-f(p);\quad \frac{p}{s}\geqslant\frac{p}{p-f(p)}=\frac{1}{1-\frac{f(p)}{p}}>1+\frac{f(p)}{p}\geqslant 1+\tau_X$$

so that

$$I_p > 2\delta Z^2 (1+\tau_X)\left(1-\frac{e}{10}\tau_X\right) > 2\delta Z^2\left(1+\frac{1}{2}\tau_X\right).$$

Introducing now

$$I'_p = I_p - \int_{-\delta}^{\delta} |S(\varkappa)|^2 \, d\varkappa \tag{6}$$

we can write

$$\int_0^1 |S(\alpha)|^2 \, d\alpha \geqslant \sum_p I'_p \tag{7}$$

the summation being extended over Y primes mentioned above. In fact, the integrand is positive and the sets, over which the integration extends, do not intersect for

$$\left|\frac{x}{p_1}-\frac{y}{p_2}\right| \geqslant \frac{1}{p_1 p_2} \geqslant \frac{1}{X} > 2\delta$$

x and y being not equal to zero. Now using the obvious inequality

$$\int_{-\delta}^{\delta} |S(\varkappa)|^2 \, d\varkappa \leqslant 2\delta Z^2,$$

we get from (6)

$$I'_p > 2\delta Z^2\left[\left(1+\frac{1}{2}\tau_X\right)-1\right] = \delta Z^2 \tau_X;$$

substituting it into (7), summing over Y primes p and using (3), we get

$$Z \geqslant Y \delta Z^2 \tau_X; \quad \delta = \frac{\tau_X}{20\pi X}.$$

Hence $Y \leqslant \dfrac{20\pi X}{\tau_X^2 Z}$, q. e. d.

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REFERENCES

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