ED 184 Mathématiques et informatique de Marseille
Institut de Mathématiques de Marseille (UMR7373)

Mémoire présenté pour obtenir
l'habilitation à diriger des recherches

Discipline : Mathématiques

Sary DRAPPEAU

Indépendance statistique et lois limites pour quelques objets arithmétiques

Statistical independence and limit laws for some arithmetical objects

Defended Septembre 28th 2023 in front of the jury composed of :

Valérie BERTHÉ	CNRS	Referee
Valentin BLOMER	Bonn Universität	Referee
Régis DE LA BRETÈCHE	Université Paris Cité	
Étienne FOUVRY	Université Paris Sud	
Florent JOUVE	Université de Bordeaux	
Kaisa MATOMÄKI	Turun yliopisto	Referee
Philippe MICHEL	École Polytechnique Fédérale de Lausanne	
Joël RIVAT	Université d'Aix-Marseille	
Emmanuel ROYER	Université Clermont-Auvergne et CNRS	

Numéro national de thèse/suffixe local : 2019AIXM0001/001ED62

Thanks

Je suis tout d'abord redevable à Joël Rivat, Étienne Fouvry et Régis de la Bretèche pour leurs subtiles incitations, leurs précieux conseils et leurs encouragements durant tout le processus de HDR.

I'm grateful to Valérie Berthé, Valentin Blomer and Kaisa Matomäki for their time, and sincerely honoured that they agreed to report on the manuscript.

Je remercie vivement Florent Jouve, Philippe Michel et Emmanuel Royer pour avoir accepté de participer au jury.

Je remercie encore Mme Tinel pour toute son aide au bureau des HDR.

Faire des mathématiques serait une activité bien morne si l'on ne pouvait la vivre avec les collègues. Je remercie donc toutes les personnes avec qui j'ai pu partager l'envie de comprendre, l'enthousiasme et l'émerveillement, que cela ait mené à un papier ou non. Je remercie en particulier Olivier Ramaré pour d'innombrables discussions éclairantes sur la théorie analytique des nombres ; et Brigitte Vallée pour avoir pris le temps de donner de nombreuses explications sur ses travaux.

Je remercie enfin mes collègues de l'I2M pour l'atmosphère conviviale, qui fait que ça vaut largement la peine de prendre le B1 en heure de pointe ou de cohabiter avec des marteaux-piqueurs.

Résumé

Ce mémoire présente les thèmes sur lesquels ont porté mes travaux de recherche depuis mon arrivée à l'université d'Aix-Marseille en 2015. Leur problématique commune est de mettre en évidence des comportements statistiques réguliers dans des familles d'objets arithmétiques naturels : les fonctions multiplicatives ou additives, et les valeurs centrales de certaines familles de fonctions L.

La première partie concerne une question centrale en théorie multiplicative des nombres : celle d'estimer la corrélation des valeurs $f(n)$ et $g(n+1)$, où f et g sont deux fonctions multiplicatives, notamment lorsque l'une des deux fonctions est la fonction « nombre de diviseurs ». Ce problème est naturellement lié à la répartition de certaines suites dans les progressions arithmétiques, et trouvent des applications à d'autres questions arithmétiques, par exemple les zéros de petite hauteur des fonctions L de Dirichlet. Les majorations de sommes d'exponentielles algébriques sont un outil crucial dans cette partie du mémoire.

La seconde partie concerne certaines fonctions $f : \mathbf{Q} \to \mathbf{C}$ nommées par Zagier « formes modulaires quantiques », caractérisées par certaines symétries analogues à celles des formes modulaires. Mes collaborations sur ce sujet ont consisté d'une part à établir ces relations de modularité quantiques dans certains cas : celui de tordues additives de fonctions L de Dirichlet, et celui de sommes de symboles de Pochhammer ; et d'autre part à les utiliser pour en déduire, par des méthodes de systèmes dynamiques, l'existence de lois limites pour les valeurs de f aux nombres rationnels ordonnés par dénominateurs croissants.

Mots clés : Fonctions multiplicatives, nombres premiers, progressions arithmétiques, sommes de Kloosterman, formes modulaires, formule de Kuznetsov, tordue additive, forme modulaire quantique, loi limite, invariant de Kashaev.

Abstract

This manuscript presents the themes of my research works in Aix-Marseille university since 2015. Their common theme is the search for simple statistical behaviour among families of natural arithmetical objects: multiplicative or additive functions, functions defined in terms of numeration systems (decimal, continued fractions...) and central values of L-functions.

The first part concerns a key question in multiplicative number theory: to estimate the correlation of values of $f(n)$ and $g(n+1)$, where f and g are two multiplicative functions, with an emphasis on the case of the divisor function. This naturally involves bounds on algebraic exponential sums, and leads to applications in various problems, all linked in some way to the distribution of certain sequences in arithmetic progressions.

The second part concerns maps $f : \mathbf{Q} \to \mathbf{C}$ called by Zagier "quantum modular forms", which satisfy certain symetries analogous to those satisfied by modular forms. In several collaborations, we established the modular quantum behaviour in some cases related to additive twists of central L values, or to Pochhammer symbols, and we deduced through methods from dynamical systems the existence of limit laws for values of f along rationals ordered by denominators.

Keywords: multiplicative functions, prime numbers, arithmetic progressions, Kloosterman sums, modular form, Kuznetsov formula, additive twist, quantum modular form, limit law, Kashaev invariant

Table des matières

Introduction

This thesis presents the results I've obtained since 2015 in the course of my research in analytic number theory. Their commun setup is to find limit laws among families of natural arithmetical objects : multiplicative or additive functions, functions defined in terms of numeration systems (decimal, continued fractions...), central values of families of L-functions.

The first part of the thesis is centered on the question of how to estimate asymptotically the correlation sum

$$C(f, g; x) := \sum_{n \leq x} f(n) g(n+1)$$

when f and g are multiplicative functions. Heuristically, we expect the factorizations of n and $n+1$ to be statistically independent on average over n. We therefore expect $C(f, g; x)$ to be comparable to the product of the averages of f and g up to x, which means

$$C(f, g; x) \asymp x^{-1} C(f, 1; x) C(1, g; x).$$

We will focus on the special case when $f(n) = \tau(n) = \sum_{d|n} 1$, the divisor function, and when the values $g(p)$ have a simple description. This case is of particular importance because it is close to the threshold past which the analytical techniques currently known are ineffective. The most significant progress on this question were made in the 1980's, which the development of techniques from automorphic forms theory to study certain algebraic exponential sums which naturally appear.

In Section 1.1, we present a generalization of results of Deshouillers and Iwaniec on the Fourier coefficients of modular forms, and bounds on averages of Kloosterman sums which follows by the Kuznetsov trace formula. These estimates are then used to deduce an arithmetical result on bounds on average over primes of Kloosterman sums under the existence of Siegel zeroes, obtained with J. Maynard.

In Section 1.2, we return to the estimation of $C(\tau, g; x)$, and we present works which answer two natural questions : how the location of zeroes of Dirichlet L-functions influence the error term in the estimation of $C(\tau, g; x)$; and a simple characterization of arithmetical functions g for which we are currently able to estimate precisely the correlation sum $C(\tau, g; x)$. The first work relies on the bounds on Kloosterman sums mentioned above. The second result, obtened with B. Topacogullari, depends on new combinatorial decompositions for certain multiplicative functions.

In Section 1.3, we present a work with K. Pratt and M. Radziwiłł in which we study the density of zeroes of Dirichlet L-functions near the real axis, on average over both the character and the modulus. The bounds for Kloosterman sums refered to above will allow us to estimate this density for test functions whose Fourier transform has a support larger than the range permitted by the large sieve inequalities.

Lastly, in Section 1.4, we present a work with R. de la Bretèche on the distribution of values of quadratic polynomials in arithmetic progressions, which also relies on the bounds for Kloosterman sums refered to above. As a corollary of this work, we improved a bit on a result of Deshouillers and Iwaniec on the largest prime divisor of $n^2 + 1$.

The second part of the thesis is centered on the existence of limit laws for certain maps naturally defined on the rationals.

The first result on this second part concerns the distribution of values of the maps $f : \mathbf{Q} \to \mathbf{C}$ such that for any homography $\gamma \in \mathrm{PSL}_2(\mathbf{Z})$, the maps $h_\gamma(x) := f(x) - f(\gamma x)$ are regular, in a certain precise sense. Such maps were called by Zagier "quantum modular forms" (of weight 0 and level 1) because some examples came from quantum invariants in knot theory. By using the Euclid algorithm and dynamical properties of the Gauss map, we show under weak hypotheses on the maps h_γ, that the values $f(x)$ for x varying among rational numbers of denominators up to Q tend to distribute along a stable low as $Q \to \infty$. This extends work of Baladi and Vallée regarding the hypotheses on h_γ. This formalism allows us to obtain a limiting law for rational values x sorted by denominators of the Estermann's function, which is the value at $s = 1/2$ of the "additive twist" of the function $\zeta(s)^2$,

$$D(s,x) := \sum_{n \geq 1} \frac{\tau(n)}{n^s} \mathrm{e}^{2\pi i n x} .$$

This series does not have an Euler product ; however, the map $x \mapsto D(1/2, x)$ is a quantum modular form for which our results apply. The method involves mostly arguments from dynamical systems, and the arithmetic is only invoked to prove the quantum modularity of the map $x \mapsto D(\frac{1}{2}, x)$.

Our second result concerns the q-analogue of the Pochhammer symbol,

$$(q)_n = \prod_{r=1}^{n} (1 - q^r),$$

when q is a root of unity. We show that the maps which to $x \in \mathbf{Q}$ associate respectively $(\mathrm{e}(x))_n$ and $(\mathrm{e}(\bar{x}))_n$ (where $\overline{h/k} = (h^{-1} \mod k)/k \pmod 1$) each satisfies, for $\gamma \in PSL_2(\mathbf{Z})$, a form of modularity which relates their values at x and γx.

These two formulas are close relatives of the modularity of the Dedekind η function. These formulas are consistent with the Zagier modularity conjecture on Kashaev invariants of hyperbolic knots. We show, using these formulas, that the Kashaev invariatns of the figure-eight knot

$$J_{4_1,0}(x) = \sum_{n\geq 0} |(\mathrm{e}(x))_n|^2$$

have, for asymptotically almost all rational x (in a precise sense), an asymptotic behaviour which can be expressed in simple terms of the continued fraction expansion of x.

1 Exponential sums and statistical independence of arithmetic functions

1.1 Kloosterman sums

1.1.1 Bounds on average

For all $q \in \mathbf{N}_{>0}$ and $a, b \in \mathbf{Z}/q\mathbf{Z}$, define the *Kloosterman sum* (Kloosterman 1927) as

$$S(a,b;q) := \sum_{n \in (\mathbf{Z}/q\mathbf{Z})^\times} \mathrm{e}\left(\frac{an + bn^{-1}}{q}\right)$$

where $\mathrm{e}(x) := \mathrm{e}^{2\pi i x}$, and n^{-1} denotes an inverse of n modulo q. We obviously have the trivial bound

$$|S(a,b;q)| \leq \varphi(q).$$

Any improvement on this bound says something on the pseudorandomness of the sequence (n^{-1}) as n varies in $(\mathbf{Z}/q\mathbf{Z})^\times$. In full generality, we have the Weil bound (Weil 1948) : for any $\varepsilon > 0$,

$$|S(a,b;q)| \ll_\varepsilon q^{1/2+\varepsilon}, \qquad ((a,b,q) = 1)$$

where (a,b,q) denotes the gcd. The exponent $1/2$ is optimal. If a, b are non-zero and fixed, we expect to be able to reach additionnal compensations in sums over q of $S(a,b,q)$. The first result of this type is due to Kuznetsov (Kuznetsov 1981), and asserts that for any $\varepsilon > 0$ and $(a,b) \in \mathbf{Z}^2 \setminus 0$,

$$\left| \sum_{q \leq x} \frac{S(a,b;q)}{q} \right| \ll_{\varepsilon,a,b} x^{1/6+\varepsilon}.$$

This path has been developed in a systematic way by Deshouillers and Iwaniec (Deshouillers ; Iwaniec 1982a) (see also (Iwaniec 1982)). These ideas have prospered and contributed to some of the most precise results in modern analytic number theory (Bombieri ; Friedlander ; Iwaniec 1986 ; Young 2011 ; Pitt 2013 ; Bettin ; Bui ; Li et al. 2020 ; Pratt ; Robles ; Zaharescu et al. 2020 ; Assing ; Blomer ;

Li 2021 ; Maynard 2020).

The results of Deshouillers-Iwaniec (Deshouillers ; Iwaniec 1982a) are about exponential sums of the following shape :

$$\sum_n \sum_c \sum_d \sum_r \sum_s \beta_{n,r,s} g(c,d) \, \mathrm{e}\Big(n\frac{(sc)^{-1}}{rd}\Big),$$

where $(\beta_{n,r,s})$ is a finitely supported sequence, and g is a smooth compactly supported function. The case when $\beta_{n,r,s} = 1$ if $n = r = s = 1$ and 0 otherwise, which means that the sums runs in effect only over c and d, corresponds to the Kloosterman sums case considered by Kuznetsov (Kuznetsov 1981), upon using Poisson summation on the variable d. The quintuple sum above is a natural generalization to a realistic setup, where the "smooth" variables c and d are perturbed by multiplicative convolution.

The following result generalizes Theorem 12 of (Deshouillers ; Iwaniec 1982a). It was proved in (Drappeau 2017), and then made more precise in (Drappeau ; Pratt ; Radziwiłł 2022).

Théorème 1 (Drappeau 2017)**.** *Let $C, D, N, R, S \geq 1$ and $q, c_0, d_0 \in \mathbf{N}$ with $(q, c_0d_0) = 1$. Let $(b_{n,r,s})$ be a sequence supported in $[1, N] \times [R, 2R] \times [S, 2S] \cap \mathbf{N}^3$. Let $g : \mathbf{R}_+^5 \to \mathbf{C}$ be a smooth function compactly supported in $[C, 2C] \times [D, 2D] \times (\mathbf{R}_+^*)^3$ which satisfies the bound*

$$\frac{\partial^{\nu_1+\nu_2+\nu_3+\nu_4+\nu_5} g}{\partial c^{\nu_1}\partial d^{\nu_2}\partial n^{\nu_3}\partial r^{\nu_4}\partial s^{\nu_5}}(c,d,n,r,s) \ll_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} \{c^{-\nu_1}d^{-\nu_2}n^{-\nu_3}r^{-\nu_4}s^{-\nu_5}\}^{1-\varepsilon_0} \quad (1.1)$$

for some $\varepsilon_0 > 0$ and all $\nu_j \geq 0$. Then

$$\sum_{\substack{c \\ c\equiv c_0}} \sum_{\substack{d \\ \text{et } d\equiv d_0 \\ (qrd,sc)=1}} \sum_{\substack{n \\ \bmod q}} \sum_r \sum_s b_{n,r,s} g(c,d,n,r,s) \, \mathrm{e}\Big(n\frac{r\bar{d}}{sc}\Big)$$
$$\ll_{\varepsilon,\varepsilon_0} (qCDNRS)^{\varepsilon+O(\varepsilon_0)} q^{3/2} K(C,D,N,R,S) \|b_{N,R,S}\|_2,$$

where $\|b_{N,R,S}\|_2^2 = \sum_{n,r,s} |b_{n,r,s}|^2$, and

$$\begin{aligned} K(C,D,N,R,S)^2 &= qCS(RS+N)(C+RD) \\ &\quad + C^{1+4\theta}DS((RS+N)R)^{1-2\theta}(1+\tfrac{qC}{RD})^{1-4\theta} + D^2NR. \end{aligned} \quad (1.2)$$

Note that in a recent paper, Assing, Blomer and Li (Assing ; Blomer ; Li 2021) prove a result which generalizes Theorem 12 of (Deshouillers ; Iwaniec 1982a) in another direction, linked to the uniformity in n.

The main difference between theorem 1 and (Deshouillers ; Iwaniec 1982a, Theorem 12) is the fact that the variables c, d, attached to a smooth weight, can be restricted to arithmetical progressions. The main idea which makes this possible was introduced by Blomer and Milićević (Blomer ; Miličević 2015) : it is to exploit the possibility of varying the central character of the automorphic forms for $\Gamma_0(q)$ underlying the analysis in (Deshouillers ; Iwaniec 1982a). The success of the method relies on a suitable choice of Poincaré series, the Fourier expansion of which around well-chosen cusps will spawn the desired Kloosterman sums. The usefullness of this setup was confirmed in many recent works on modular L functions, where the Kuznetsov formula plays a rôle : let us cite (Kıral ; Young 2019), (Petrow ; Young 2020) and (Zacharias 2019).

A difference which is less essential is the dependence in θ, which is made explicit, whereas the bound $\theta \leq 1/4$ from Selberg was used in (Deshouillers ; Iwaniec 1982a). This only affects the second term in (1.2). In most of the applications, this term is not the limiting one, but this is not systematically the case, as we will see in theorem 6 below.

Due to bounds of the type of theorem 1, Kloosterman sums play a prominent role in analytic number theory. It would be very desirable to have at our disposal bounds of the same quality for other exponential sums, such as Birch sums, hyper-Kloosterman sums, or exponential sums whose phase parametrizes solutions to $n^3 \equiv 2 \mod p$. Recent works on such a setup was done by Buttcane, see (Buttcane 2022).

1.1.2 Average value along prime numbers and Siegel zeroes

The work of Blomer and Milićević (Blomer ; Miličević 2015), which we refered to above, allows one to bound the sums

$$\sum_{q \leq x} \chi(q) S(a, b; q), \tag{1.3}$$

in which χ is a Dirichlet character. It is natural to wonder if the extra flexibility allowed by the character χ could give additional arithmetic applications We would be particularly happy to be able to prove the following statement,

$$\sum_{p \leq x} \frac{S(a, b; p)}{\sqrt{p}} \stackrel{?}{=} o(x/\log x), \tag{1.4}$$

where p denotes a prime number.

The question (1.4) is still wide open. It would follow from the “horizontal” Satō-Tate conjecture, made by Katz (Katz 1980). In 2007, Fouvry and Michel (Fouvry ; Michel 2007) have obtained a non-trivial bound when the primality condition is relaxed to the condition that n belongs to the set P_k of integers having at most k prime divisors, for some fixed k. Their method has been the subject of many improvement, and the best known value of an admissible k is $k = 7$ by Xi (Xi 2018).

The possibility to exploit the dependence in the character χ in (1.3) allows for an application, should there exist a Dirichlet character which takes the value -1 abnormally often on prime numbers. More precisely, we know that there is a constant $c > 0$ such that for all $Q \geq 2$ and each Dirichlet character χ primitive modulo $q \leq Q$, the inequality

$$L(1, \chi) \geq c/ \log q$$

holds with the exception of at most one such character χ, which is then quadratic. We conjecture that no such exceptions exist, and in fact the Riemann hypothesis generalized to Dirichlet L functions implies the inequality $L(1, \chi) \gg 1/ \log\log q$; but no substantial progress has been made on this question since it arose. In full generality, the Siegel bound $L(1, \chi) \gg_\varepsilon q^{-\varepsilon}$ for all $\varepsilon > 0$, which is non-effective as soon as $\varepsilon < 1/2$, is the best-known one.

Suppose there is a sequence of primitive and quadratic Dirichlet character $\chi_n \pmod{D_n}$ such that $L(1, \chi_n) \log D_n \to 0$ as $n \to \infty$. These characters would then give a family of multiplicative functions which “mimick” the Möbius function on most integers on size $D_n^{O(1)}$ without large prime divisors, while at the same time, being of a controlled analytic complexity, since χ_n is periodic modulo D_n.

This idea was pursued in a seminal work of Heath-Brown in 1983 (Heath-Brown 1983), and in many recent works of Friedlander and Iwaniec (Friedlander ; Iwaniec 2004 ; Friedlander ; Iwaniec 2005 ; Friedlander ; Iwaniec 2013). It follows from these works that the existence of Siegel characters would allow to solve many problems which are just beyond the threshold of modern analytic techniques : the twin prime conjecture (Heath-Brown 1983), or primes in arithmetic progressions of large moduli (Friedlander ; Iwaniec 2003), or in very short intervals (Friedlander ; Iwaniec 2004).

With James Maynard, we have proved in (Drappeau ; Maynard 2019) that the existence of exceptional characters would imply a non-trivial bound for the average value of Kloosterman sums along prime numbers.

Théorème 2 (Drappeau ; Maynard 2019). *For any $\varepsilon > 0$, there exist $A, B > 0$ such that for any primitive quadratic character $\chi \pmod{D}$ and all $x \geq D^A$, we have*

$$\Bigg| \sum_{p \leq x} \frac{S(1,1;p)}{\sqrt{p}} \Bigg| \leq \pi(x) \Big(\varepsilon + B L(1, \chi) \log x \Big).$$

Since $|S(1,1;p)| \leq 2\sqrt{p}$ by the Weil bound, the above statement is non-trivial as soon as there exists a sequence $\chi_n \pmod{D_n}$ of primitive quadratic characters which satisfies $L(1, \chi_n) \log D_n \to 0$, in other words, under the existence of Siegel characters. The estimate from theorem 2 is then relevant in a range of the shape $[D^A, D^C]$ with C arbitrary but fixed.

The basic strategy is the one followed by Heath-Brown (Heath-Brown 1983) and Friedlander-Iwaniec (Friedlander; Iwaniec 2005; Friedlander; Iwaniec 2004), which consists in approaching the van Mangoldt function $\Lambda(n) = \log * \mu(n)$ by the convolution $\log * \chi(n)$. From the point of view of analytic complexity, this last function is comparable to $\mathbf{1} * \mathbf{1}(n)$, the divisor function. The problem then translates to

$$\sum_{N<n\leq 2N} \sum_{M<m\leq 2M} S(1,1,mn).$$

When $M < N^{1-\varepsilon}$ ou $N < M^{1-\varepsilon}$, the estimates from Deshouillers-Iwaniec, which are of the same kind of theorem 1, give an acceptable bound. The main difficulty in the method is the case $M = N$. In this case, we are not able to exploit cancellation in the sign of the Kloosterman sums. The crucial idea is to replace $\log(n/d)$ in the convolution $\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d)$, by $\log(\sqrt{n}/d)$. This has strictly no effect on the rest of the arguments, but has the advantage of dampening the effect of those d of size about $\sqrt{n}$, which correspond to the problematic case above. The rest of the argument follows a method of Fouvry-Michel (Fouvry; Michel 2003), based on an idea of Hooley (Hooley 1964) and the numerical coincidence $8 < 3\pi$, which shows that $|S(1,1,n)|$ becomes negligible, on average over n, as n has more and more prime divisors. This idea arises in many other works on Kloosterman sums, notably the recent work of Xi (Xi 2020) on the non-coincidence between Kloosterman sums and Hecke-Maass eigenvalues.

1.2 Shifted convolution with the divisor function

1.2.1 The error term in the Titchmarsh divisor problem

Bounds on exponential sums of the same type as theorem 1 naturally find applications to the estimation of "shifted convolutions"

$$\sum_{n\leq x} f(n)g(n+1), \tag{1.5}$$

where f and g are assumed to be multiplicative, or to have a strong link with a multiplicative functions, such as the characteristic function of prime numbers.

One approach in this problem is to approximate f and g by non-trivial Dirichlet convolutions, which amounts to factorizing $n = ab$ and $n+1 = cd$, and then to count solutions to the equation $cd - ab = 1$ weighted by some coefficients (Duke; Friedlander; Iwaniec 1994). In the best case scenario, the coefficients vary smoothly with a, b, c, d, and by Fourier analysis, we get back to exponential sums of the same shape as Kloosterman sums. This explains why estimates based on Kloosterman sums are particularly interesting in multiplicative number theory. We refer to (Kowalski 2003) for a closely related discussion.

The divisor function $\tau = \mathbf{1} * \mathbf{1}$ can be seen as the most simple arithmetic function which is not a perturbation of a constant function. The higher-order divisor functions $\tau_k = \mathbf{1} * \cdots * \mathbf{1}$ (k times) are the most natural generalizations. The indicator function of primes $n \mapsto \mathbf{1}_{\mathcal{P}}(n)$ can be seen as a sort of limit of the maps τ_k, in the sense that if a question concerning a map f can be resolved for $f = \tau_k$, k arbitrary, with some uniformity in k, we expect that the same question could be resolved for $f = \mathbf{1}_{\mathcal{P}}$. This vague classification of multiplicative functions materializes in “combinatorial identities” for multiplicative functions, the most well-known of which are those of Linnik, Vaughan and Heath-Brown, and which will be the topic of this section.

The Titchmarsh divisor problem (Titchmarsh 1930) consists in estimating the sum

$$T(x) := \sum_{p \leq x} \tau(p-1).$$

It is a special case of the question (1.5). The best estimation known is

$$T(x) = c_1 x + c_2 \operatorname{li}(x) + O\Big(\frac{x}{(\log x)^A}\Big) \tag{1.6}$$

due independently to Fouvry (Fouvry 1985) and Bombieri-Friedlander-Iwaniec (Bombieri ; Friedlander ; Iwaniec 1986). The shape of the error term follows from the use of the prime number theorem in arithmetic progressions, and the Siegel-Walfisz theorem. It is a natural question to ask if a hypothesis such as Riemann’s hypothesis on the ζ function would allow one to have a power-saving error term in x, as is the case in the prime number theorem. The following result, obtened in (Drappeau 2017), shows that the answer is yes.

Théorème 3 (Drappeau 2017). *Assume the Riemann hypothesis generalized to all Dirichlet L functions. Then there exists $\delta > 0$ such that*

$$T(x) = c_1 x + c_2 \operatorname{li}(x) + O(x^{1-\delta}) \qquad (x \geq 2).$$

This result is actually the one which originally motivated theorem 1. An effective value of δ was obtained by Tang (Tang 2020). It is natural to expect that the optimal error term should be $O(x^{1/2+\varepsilon})$, but the best value of δ permitted by the method is quite far away from that.

The strategy is very close to the work of Fouvry (Fouvry 1985) and Bombieri-Friedlander-Iwaniec (Bombieri ; Friedlander ; Iwaniec 1986), but the constraint to obtain a power-saving error term forbids us to use certain techniques they use, from sieve theory notably. By taking these constraints into account, it transpires that the exponential sum one has to estimate has the shape

$$\sum_{c,d,n,r,s} b_{n,r,s} g(c,d) \operatorname{e}\left(\frac{n(sc)^{-1}}{rd} + \frac{(cd)^{-1}}{q}\right) \tag{1.7}$$

where $q \geq 1$ is a relatively small integer, of the order of x^ε. Previous works (Fouvry 1985 ; Bombieri ; Friedlander ; Iwaniec 1986) used several reductions to restrict to the case $q = 1$, which brought the above sum into the range which Deshouillers-Iwaniec's work (Deshouillers ; Iwaniec 1982a) could handle. But these reductions are made at a cost of the order of $x\,\mathrm{e}^{-c\sqrt{\log x}}$ at best. However, having theorem 1 at our disposal, we may work directly with the original sum (1.7) by separating the variables c and d according to their congruence classes modulo q, keeping power-saving error terms in x throughout the argument. This leads to the proof of theorem 3.

1.2.2 Titchmarsh's problem for other multiplicative functions

Consider the sum (1.5) with $f = \tau$. We know how to obtain an asymptotic estimate for $g = \mathbf{1}_{\mathcal{P}}$, as we have seen in the previous paragraph. However, until 2017, nothing seemed to be known for functions which are considered to be of the same difficulty, or simpler, such as $g = \mathbf{1}_{\mathcal{B}}$, the indicator function of the set $\mathcal{B}$ of integers which are sums of two squares.

More generally, we expect to be able to handle the case of the map $g = \tau_z$ for $z \in \mathbf{C}$ in the Titchmarsh divisor problem, because it is considered to be of the same "analytic difficulty" as the indicator function of primes : for instance, the map $s \mapsto \zeta(s)^z$ possesses singularities at zeroes of the Riemann function ζ, which are of logarithmic nature.

We resolved this problem in a work with Berke Topacogullari (Drappeau ; Topacogullari 2019). The "analytic difficulty" which we mentioned above materializes, in our result, through a hypothesis that the function g behaves as a periodic function when restricted to prime numbers.

Théorème 4 (Drappeau ; Topacogullari 2019)**.** *Suppose that the multiplicative function $g : \mathbf{N} \to \mathbf{C}$ satisfies the following hypotheses :*

- *For some $D \in \mathbf{N}$, whenever p, q are prime numbers with $p \equiv q \pmod{D}$, then $g(p) = g(q)$.*
- *For some $A > 0$, we have $|g(n)| \leq \tau(n)^A$ for all $n \in \mathbf{N}$.*

Then for all $B > 0$, we have

$$\sum_{1<n\leq x} g(n)\tau(n-1) = 2 \sum_{\substack{\chi \text{ primitive} \\ \mathrm{cond}(\chi)|D}} \sum_{\substack{q\leq\sqrt{x} \\ \mathrm{cond}(\chi)|q}} \frac{1}{\varphi(q)} \sum_{\substack{q^2\leq n\leq x \\ (n,q)=1}} g(n)\chi(n) + O_{D,A,B}(x/(\log x)^B). \tag{1.8}$$

The sum over n in the right-hand side is easy to estimate by using classical facts on the zero-free region of Dirichlet L-functions. Fouvry and Tenenbaum (Fouvry ; Tenenbaum 2022) have recently obtained a result which contains theorem (1.8) and yields new applications on the shifted convolution problem (1.5) for additive functions f, g.

Compared with the arguments underlying the Titchmarsh problem (1.6), the part which we improve upon is the one commonly referred to as the "combinatorial identity". It is an argument which has been studied extensively for prime numbers (Vinogradov, Linnik, Gallagher, Vaughan, Heath-Brown) ; we refer to the survey paper by Ramaré (Ramaré 2013) for a much more detailed discussion of the history of these techniques. The new ingredient in theorem 4 is a combinatorial identity for a multiplicative function g satisfying the hypotheses of theorem 4. In fact we obtain two such identities.

The first is an identity of Heath-Brown's type (Heath-Brown 1982) for τ_α with $\alpha \in \mathbf{Q}$. The possibility that such a formula exists was suggested by another identity, due to Robert C. Vaughan, for $\tau_{1/2}$ which was kindly communicated to us by Hugh Montgomery thanks to Olivier Ramaré. This identity for $\tau_{1/2}$, which we do not display here, will appear in volume II of *Multiplicative Number Theory*, but we have not managed to use it in the Titchmarsh problem. After some trial-and-error, we finally found a suitable generalization of Heath-Brown's identity. To continue with the simple special case of the function $\tau_{1/2}$, it has the shape

$$\sum_{m\geq 4} a_m(\zeta(s)^{1/2}M_x(s)-1)^m\zeta(s)^{1/2} = \zeta(s)^{1/2} + \sum_{\ell\geq 1} b_\ell \zeta(s)^\ell M_x(s)^{2\ell-1}, \tag{1.9}$$

where (am), (b_ℓ) are complex numbers, ζ is the Riemann zeta function, and

$$M_x(s) = \sum_{n\leq x^{1/4}} \tau_{-1/2}(n)n^{-s}$$

is an approximation to $\zeta(s)^{-1/2}$. The Dirichlet series on the left-hand side of (1.9) has no terms of index $\leq x$, while the series on the right-hand side contains $\zeta^{1/2}$ on the one hand, and non-negative integer powers of ζ and of the truncated series $M_x g$ on the other hand. The coefficients a_m and b_ℓ are obtained by first finding a polynomial of the shape $P(X) = 1 + XQ(X^2)$, $Q \in \mathbf{C}[X]$, which vanishes at 1 at order 4, for instance $P(X) = 1 - \frac{35}{16}X + \frac{35}{16}X^3 - \frac{21}{16}X^5 + \frac{5}{16}X^7$. We then formally substitute $X = \zeta^{1/2}M$ in the equality $P(X) = 1 + XQ(X^2)$, and then we multiply $\zeta^{1/2}$.

It is much less obvious to see that this method generalizes well to the function τ_α for $\alpha = u/v$ rational, with a good control in terms of v on the size of the coefficients b_ℓ. This last point is crucial. We prove that such is indeed the case ; the size of b_ℓ is then essentially controlled by the (archimedean) size of $|u/v|$. This allows us, as a first step, to estimate the sum

$$\sum_{n\leq x} \tau_\alpha(n)\tau(n+1)$$

uniformly in $\alpha \in \mathbf{Q}$, $|\alpha| \ll 1$, $\mathrm{denom}(\alpha) \ll (\log x)^A$, with an error term of size $O(x/(\log x)^A)$.

To tackle the general case τ_z, $z \in \mathbf{C}$, $|z| \ll 1$, the second idea we use is simply a Lagrange interpolation on the polynomial map

$$z \mapsto \sum_{n \leq x} \tau_z(n)\tau(n+1).$$

The simplicity of this approach hides, of course, the issue to control the dependency in x in the interpolation step. We first estimate the contribution of terms of degree $\geq C \log\log x$ by $O_C(x/(\log x)^{C+O_z(1)})$ for $C \geq 7$.[1] The order of magnitude $\log\log x$ is related to the typical value of the number of prime divisors of integers of size up to x (a theorem of Hardy-Ramanujan). Then, a natural choice of sampling points consisting of rational numbers of denominators $O(\log\log x)$, in the Lagrange interpolation step, leads to an error term of the shape

$$[\text{error term at the sampling points}] \times \exp([\text{degree}]), \tag{1.10}$$

which is to say $\frac{x}{(\log x)^A} \times \exp(C \log\log x)$, with A arbitrarily large. Finally, we obtain for all z an estimate with error term

$$O_{A,C}\left(\frac{x}{(\log x)^{C+O_z(1)}} + \frac{x}{(\log x)^{A-C}}\right),$$

and a suitable choice of A and C concludes the argument. The success of this approach crucially relies on the dependence on the degree in (1.10).

The second proof we have obtained is very similar to Vinogradov's original works (Vinogradov 1937). It does not rely on an interpolation step, and is therefore somewhat simpler to set up. The starting point is the Linnik identity (Linnik 1963, Chapitre VIII), which is to write

$$\zeta^z = (1 + (\zeta - 1))^z = \sum_{j \geq 0} \binom{z}{j} (\zeta - 1)^j.$$

As we expand the right-hand side as a Dirichlet series, the coefficient of an integer n involves indices j of size $O(\log n)$, which is way too large in practice. The idea now is to factor the y-friable part of integers at the level of Dirichlet series, which is to say, to write

$$\zeta = \zeta_y M_y, \qquad \zeta_y(s) = \prod_{p \leq y} (1 - p^{-s})^{-1}, \quad M_y(s) = \prod_{p > y} (1 - p^{-s})^{-1}.$$

We then chose $y = x^\varepsilon$ for some small value of $\varepsilon > 0$ to be determined. By using

1. The exponent is not asymptotically optimal but it suffices for us that it tends to infinity with C.

the strategy of Linnik to the part with M_y, we obtain

$$M_y^z = \sum_{j\geq 0} \binom{z}{j} (M_y - 1)^j,$$

and if we wish to detect the coefficient of an integer $n \leq x$, the sum over j can be truncated at $j < 1/\varepsilon$. To conclude, we bring back the various factors ζ_y, which has no impact on the argument as long as ε is sufficiently small. In practice $\varepsilon < 1/4$ does the job. This is due to the fact that the indicator function of friable integers benefits from good factorization properties in the sense of Dirichlet convolution. In the case of the divisor function, we obtain for instance the decomposition

$$\tau_z(n) = \sum_{0\leq \ell \leq 3} c_{\ell,z} \sum_{\substack{n=n_1 n_2 \\ p|n_1 \implies p\leq x^{1/4}}} \tau_{z-\ell}(n_1)\tau_\ell(n_2), \qquad (n \leq x), \tag{1.11}$$

where

$$\begin{aligned} c_{0,z} &= 1 - \tfrac{11}{6}z + z^2 - \tfrac{1}{6}z^3, & c_{1,z} &= 3z - \tfrac{5}{2}z^2 + \tfrac{1}{2}z^3, \\ c_{2,z} &= -\tfrac{3}{2}z + 2z^2 - \tfrac{1}{2}z^3, & c_{3,z} &= \tfrac{1}{3}z - \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3. \end{aligned}$$

Starting from the formula (1.11), it is relatively simple to follow the arguments of Bombieri-Fouvry-Friedlander-Iwaniec. We check that $\sum_\ell c_{\ell,z} = 1$ (which is consistent with (1.11) at $n = 1$) and that $\sum_\ell \ell c_{\ell,z} = z$ (which is consistent with (1.11) when $n = p > x^{1/4}$).

The uniformity with respect to the complex variable z allows us to use the Cauchy formula to obtain asymptotic formula relative to integers having exactly k prime divisors.

Corollaire 1 (Drappeau ; Topacogullari 2019)**.** *For $1 \leq k \ll \log\log x$, we have*

$$\sum_{\substack{n\leq x \\ \omega(n)=k}} \tau(n-1) = \frac{x}{\log x} \sum_{j=1}^{J} \frac{P_{j,k}(\log\log x)}{(\log x)^j} + O_{J,\varepsilon}\Big(x \frac{(\log\log x)^k}{k!(\log)^{J-1+\varepsilon}} \Big),$$

where for all $j \geq 1$, $P_{j,k}$ is a polynomial of degree at most $k-1$.

For $k = 1$, we recover the theorem of Bombieri-Fouvry-Friedlander-Iwaniec with an error term of the same quality. Note that the number of integers $n \leq x$ such that $\omega(n) > C\log\log x$ is $\ll_C x(\log x)^{C-1-C\log C}$. The condition on k in corollary 1 is therefore natural, since it corresponds to the domain in which the error term, which arises from an application of the Siegel-Walfisz theorem, is relevant.

The case $z = 1/2$ in (1.11) allowed us to answer the question which initially motivated us.

Corollaire 2 (Drappeau ; Topacogullari 2019). *If $\mathcal{B}$ denotes the set of numbers which can be written as sums of two squares, then for some sequence of numbers $(\beta_j)_{j\geq 0}$ and all $J \geq 0$, we have*

$$\sum_{n\in\mathcal{B}\cap[2,x]} \tau(n-1) = x(\log x)^{1/2} \sum_{j=0}^{J-1} \frac{\beta_j}{(\log x)^j} + O_J(x(\log x)^{1/2-J}).$$

These results can still be applied if $n-1$ is replaced by $n-h$ with $h \in \mathbf{Z}$, $h \neq 0$ and $|h| \leq x^\delta$ for some $\delta > 0$.

In this topic, one would like to be able to evaluate the left-hand side of (1.8) even in a situation when the mean value of f is negligible with respect to any negative power of $\log x$. The error term given in theorem 4 is then no longer negligible with respect to the main term. The case of friable integers was treated in (Fouvry ; Tenenbaum 1990 ; Drappeau 2015), in particular thanks to work of Harper (Harper 2012) for small values of y. It is natural to ask the same question for the set of integers which have exactly k prime divisors when $k/\log\log x \to \infty$, that is to say, an analogue of corollary 1 for these large values of k. The characteristic function of these integers enjoys good factorization properties, and it is reasonable to expect that some of the techniques used in (Drappeau 2015) could be adapted to treat this case. The problem of replacing the function τ in the left-hand side of (1.8) by a more general function seems, on the contrary, currently out of reach. The simplest case after τ is that of the ternary divisor function τ_3 ; the asymptotic estimation of $\sum_{n\leq x} \tau_3(n)\tau_3(n+1)$ is a major open question in analytic number theory, and essentially no approach has been remotely effective up to now.

1.3 Zeroes of Dirichlet L-functions

In this section, we assume the truth of the Riemann hypothesis generalized to Dirichlet L-functions.

The problem we discuss here concerns the vertical distribution of zeroes of Dirichlet L-functions. Let χ be a primitive Dirichlet character. We are interested in the multiset of zeroes of its associated L-function in the critical line,

$$Z_\chi := \{\gamma \in \mathbf{R}, L(\tfrac{1}{2} + i\gamma, \chi) = 0\},$$

and then to the multisets

$$Z_q := \bigcup_{\substack{\chi \pmod q \\ \text{primitif}}} Z_\chi,$$

$$Z_{\leq Q} := \bigcup_{q\leq Q} Z_q.$$

Using methods due essentially to Riemann and Weyl, it is known that

$$\operatorname{card}\{\gamma \in Z_\chi \cap [0,T]\} \sim \frac{T}{2\pi}\log\left(\frac{q}{2\pi}\right), \qquad (T \to \infty, T \le q^{o(1)}).$$

Using this estimate, it is natural to expect to find, at a distance $\asymp 1$ from the real axis, of the order of $\log q$ elements of Z_χ. Very little is known for a given fixed character χ, as $q \to \infty$. However, we have precise conjectures when we let χ vary among all Dirichlet characters modulo q, that is to say, if we average over Z_q.

Given a map $\phi : \mathbf{R} \to \mathbf{C}$ and an integer $q \ge 1$, we are interested in the quantity

$$W(\phi, q) := \sum_{\substack{\chi \pmod q \\ \text{primitif}}} \sum_{\gamma \in Z_\chi} \phi\Big(\frac{\gamma}{2\pi}\log q\Big).$$

Denote by $\psi(q)$ the number of primitive characters modulo q.

Conjecture 1. *Suppose that the Fourier transform $\hat{\phi}$ is of compact support. Then we have*

$$W(\phi, q) \sim \psi(q) \int \phi(\xi)\, \mathrm{d}\xi \tag{1.12}$$

as soon as the right-hand side tends to ∞.

This conjecture is a special case of a vast net of conjectures made by Katz-Sarnak (Katz ; Sarnak 1999), which give a hypothetical link with the distribution of eigenvalues of random matrices. These conjectures find their roots in an article of Montgomery (Montgomery 1973) on the pair correlation of zeroes of the Riemann ζ function.

Conjecture 1 and its generalizations can be sometimes established under hypotheses relative to the size of the support of the Fourier transform of ϕ. The following result, which is proved for instance in (Sica 1998), follows relatively easily from the generalized Riemann hypothesis.

Théorème 5. *Under the Riemann hypothesis generalized to Dirichlet L-function, the asymptotic equivalenc (1.12) holds true as long as $\hat{\phi}$ is supported inside $]-2, 2[$.*

In joint work with Kyle Pratt and Maksym Radziwiłł (Drappeau ; Pratt ; Radziwiłł 2022), we extend the condition on the support of $\hat{\phi}$, at the cost of having an additional average over the modulus q.

Théorème 6 (Drappeau ; Pratt ; Radziwiłł 2022). *Under the Riemann hypothesis generalized to Dirichlet L-functions, we have*

$$\sum_{q \le Q} W(\phi, q) \sim \Big(\sum_{q \le Q} \psi(q)\Big) \int \phi(\xi)\, \mathrm{d}\xi$$

if $\hat{\phi}$ is supported in $]-2-\delta, 2+\delta[$ with $\delta = 50/1093$.

A similar statement holds without having to assume the Riemann hypothesis, one simply needs to replace γ by $-i(\rho - 1/2)$ in the definition of $W(\phi, q)$.

The main feature of theorem 6 is that we pass beyond the threshold 2 on the size of the support of $\hat{\phi}$, which could not be deduced in a simple way from the Riemann hypothesis as in theorem 5. The situation is analogous to that of counting primes in arithmetic progressions of modulus q, where a trivial application of the generalized Riemann hypothesis leads to a condition $q \leq x^{1/2-\varepsilon}$ for individual moduli, which we can beat unconditionally in some cases (Fouvry 1985 ; Bombieri ; Friedlander ; Iwaniec 1986) by performing an additional average over q. This analogy is more than just formal, since the same tools are at play in both cases.

The method begins with an application of the explicit formula of Riemann-Weyl, which takes essentially the shape

$$\sum_{q \leq Q} W(\phi, q) \approx \Big(\sum_{q \leq Q} \psi(q) \Big) \int \phi(\xi) \,\mathrm{d}\xi + \sum_{q \leq Q} \sum_{\substack{\chi \pmod q \\ \text{primitif}}} \sum_{p} \chi(p) \hat{\phi}\Big(\frac{\log p}{\log q}\Big).$$

By orthogonality of Dirichlet characters, this last sum can be evaluated if we know how to estimate

$$\sum_{q \leq Q} \psi(q) \sum_{p \equiv 1 \pmod q} \hat{\phi}\Big(\frac{\log p}{\log q}\Big).$$

We are then reduced to counting primes in arithmetic progressions. The fundamental problem is the following : prove that there exist real numbers $\kappa > 2$ and $\eta > 0$ such that we have

$$\sum_{q \leq Q} \Big(\sum_{\substack{p \leq X \\ p \equiv 1 \pmod q}} 1 - \frac{\mathrm{li}(X)}{\varphi(q)} \Big) \ll Q^{1-\eta} \sqrt{X}, \qquad \text{lorsque } X = Q^{\kappa}. \tag{1.13}$$

This question may seem paradoxical. Indeed, in the estimates of Bombieri-Vinogradov's type, the range $X > Q^{2+\varepsilon}$ is the easy one, and the whole problem is to study small values of X, whereas here we wish to take X somewhat larger than Q^2. The main thing at stake is in fact the error term, which we wish to show is negligible with respect to $Q\sqrt{X}$, whereas to study prime numbers, the bound to be beaten is rather of size X. Then, for X somewhat larger than Q^2, the error term we wish to achieve is of size $X^{1-\eta}$, whereas the Bombieri-Vinogradov theorem gives us an upper-bound of size $X/(\log X)^A$.

The "proof of concept" of theorem 6 is given by the estimate of theorem 3.

This last theorem proves that if the generalized Riemann hypothesis is true, then

$$\sum_{q\le Q}\Big(\sum_{\substack{p\le X\\ p\equiv 1 \pmod q}} 1 - \frac{\mathrm{li}(X)}{\phi(q)}\Big) \ll X^{1-\eta} \qquad (X=Q^2)$$

for some $\eta>0$, which is a bound of type (1.13) for $\kappa=2$. All the work consists in showing that one may take κ slightly larger while keeping an error term of the same quality.

The fact that we can make do without the generalized Riemann hypothesis in theorem 6 is related to the fact that the characters χ which are involved in the definition of $W(\phi,q)$ are all primitive. We can therefore take away, in our whole argument, the contribution coming from characters of conductor, say, at most x^δ for some small $\delta>0$. These characters are the only obstruction to getting a power-saving error term in X in the absence of a Riemann hypothesis.

1.4 Friable values of quadratic polynomials

Another emblematic application of the estimates of Deshouillers and Iwaniec (Deshouillers ; Iwaniec 1982a) concerns the level of distribution of quadratic polynomials. This is due to the fact that the roots of a quadratic polynomial naturally give rise to an exponential sum of the same type as Kloosterman sums.

The problem is to estimate the sum

$$\Delta(x,Q,\lambda)=\sum_{q\le Q}\lambda_q\Big(\sum_{\substack{n\le x\\ n^2\equiv -1 \pmod q}} 1 - \mathrm{TP}(x,q)\Big),$$

where (λ_q) is a sequence of complex numbers of moduli at most 1, and $\mathrm{TP}(x,q)$ is the expected main term, which means

$$\mathrm{TP}(x,q):=\frac{x\rho(q)}{q}, \qquad \rho(q)=\mathrm{card}\{m \pmod q, m^2\equiv -1 \pmod q\}.$$

We may conjecture that for all $\varepsilon>0$, there exists $\delta>0$ such that

$$\Delta(x,Q,\lambda)=O(x^{1-\delta}), \qquad (Q\le x^{2-\varepsilon}),$$

uniformly in λ. In practice, we have $\sum_{q\le Q}\mathrm{TP}(x,q)\ll x\log(2Q)$, and any bound on $\Delta(x,q,\lambda)$ which is negligible with respect to $x\log Q$ is interesting. This problem is more and more difficult as the value of Q gets larger and larger.

The following result was obtained in joint work with Régis de la Bretèche (La Bretèche ; Drappeau 2020).

Théorème 7 (La Bretèche ; Drappeau 2020). *Assume that the sequence* (λ_q) *is well-factorable in the sense of Iwaniec. For all* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$|\Delta(x, Q, \lambda)| \ll x^{1-\delta} \qquad \textit{for} \quad Q \leq x^{1+25/178-\varepsilon}.$$

This improves quantitatively a result obtained by Iwaniec in 1978 (Iwaniec 1978), which dates from before “kloostermania”. Such a bound has consequences on problems involving the factorization of integers of the shape n^2+1 ; for instance, Iwaniec uses his estimate on $|\Delta(x, Q, \lambda)|$ to prove, at the outcome of significant work, that the polynomial n^2+1 takes, infinitely often, values with at most two prime factors. This is an approximation of the Landau conjecture that n^2+1 is prime for infinitely many values of n.

We applied theorem 7 to another problem, which goes back to work of Chebyshev. The question is to find values of n such that n^2+1 has a large prime divisor. The best result before 2017 was due to Deshouillers and Iwaniec (Deshouillers ; Iwaniec 1982b), who prove the lower bound

$$P^+\Big(\prod_{x\leq n<2x} (n^2+1)\Big) \gg x^{1.2024},$$

where $P^+(m)$ denotes the largest prime divisor of m, with the convention $P^+(1) = 1$. In the setting of the works which led us to theorem 7, we were able to update what was known in this topic, in the light of the progress made on the Selberg conjecture on small eigenvalues of the hyperbolic Laplacian on congruence surfaces. The explicit dependence on θ in the estimate (1.2), in particular, allows us to deduce the following result.

Théorème 8 (La Bretèche ; Drappeau 2020). *We have*

$$P^+\Big(\prod_{x\leq n<2x} (n^2+1)\Big) \gg x^{1.2182}.$$

This result was then substantially improved by Merikoski (Merikoski 2022) : we now know that

$$P^+\Big(\prod_{x\leq n<2x} (n^2+1)\Big) \gg x^{1.279}.$$

One of the new ideas in the work of Merikoski is the use of a lower bound sieve of Harman in the arguments of Deshouillers-Iwaniec, which has the effect of substantially amplifying the efficiency of bounds on exponential sums in this problem.

Another part of our work with Régis de la Bretèche (La Bretèche ; Drappeau 2020), which in fact was the original motivation for this work, is the construction of a lower bound sieve for the set of friable integers. An integer n is said to be y-friable if $P^+(n) \leq y$. Friable integer are a recurrent theme in multiplicative number theory (Granville 2008 ; Hildebrand ; Tenenbaum 1993 ; Moree 2014), and were a focus of my thesis work. Define

$$\begin{aligned} S(x,y) &:= \{n \le x, p \mid n \implies p \le y\}, \\ \Psi(x,y) &:= \operatorname{card} S(x,y). \end{aligned}$$

For all real $u \ge 1$, it is known that

$$\Psi(x, x^{1/u}) \sim x\rho(u) \qquad (x \to \infty),$$

where $\rho : \mathbf{R}_{\ge 1} \to [0,1]$ is the Dickman function. As $u \to \infty$, we have asymptotically $\rho(u) = u^{-(1+o(1))u}$. Analogously to prime numbers, we would like to be able to count asymptotically the cardinality of those integers $n \le x$ for which n^2+1 is y-friable. This can be viewed as an instance of the problem (1.5) when f is the characteristic function of y-friable integers, and g is the characteristic function of squares ; or as a “dual” version of the conjecture of Landau which we mentioned earlier. We conjecture that

$$C(x, x^{1/u}) := \operatorname{card}\{n \le x : n^2+1 \text{ is } x^{1/u}\text{-friable}\} \sim x\rho(2u),$$

on the basis of the fact that n^2+1 is typically of order of magnitude x^2, and on the heuristic that such integers, which have no fixed divisors, take friable values at a frequency similar to integers of the same size.

The question of obtaining an upper-bound on the quantity $C(x, x^{1/u})$ is simpler. The best general upper-bound, due to Khmyrova (Hmyrova 1964), yields

$$C(x, x^{1/u}) \ll x\rho(u)$$

for all $\varepsilon > 0$ and $u \ll \log x/\log\log x$, based on an elementary estimation of $\Delta(x, Q, \lambda)$ for $Q \le x^{1-\varepsilon}$. In the paper (La Bretèche ; Drappeau 2020), we construct an upper-bound sieve for friable integers, which gives, in conjuction with theorem 7, the following estimate.

Théorème 9 (La Bretèche ; Drappeau 2020)**.** *For any fixed $\varepsilon > 0$, there exists $c > 0$ such that for all $1 \le u \le c\log x/\log\log x$, we have*

$$C(x, x^{1/u}) \ll_\varepsilon x\rho((1 + \tfrac{25}{178} - \varepsilon)u).$$

As $u \asymp \log x/\log\log x$, we win asymptotically a power of x with respect to previous estimates.

Theorem 7 gives two improvements with respect to the existing literature. The first is about the numerical value of the exponent. The previous work in this topic is due to Iwaniec, which obtained an analogous estimate with exponent $1/15$, by using the Weil bound on Kloosterman sums. Using the recent works on bounds on Kloosterman sums on average, and notably theorem 1, is the main ingredient in the numerical gain. The second improvement in theorem 7 is much more substantial but also slightly more technical : contrary to earlier works using the Deshouillers-Iwaniec bounds, we have no need that the sequence (λ_q) is supported on prime numbers or squarefree numbers. This flexibility comes from an improvement in an argument of Duke-Friedlander-Iwaniec (Duke ; Friedlander ; Iwaniec 1995), and uses crucially the uniformity in q allowed in theorem 1.

2 Modularity, continued fractions and limit laws

In this second part, the results we present concern objects originating from arithmetic, which are studied using methods from dynamical systems.

We first describe the problem which initiated the works presented in this part. It is about the series

$$D(s,x) := \sum_{n\geq 1} \frac{\tau(n)}{n^s} \mathrm{e}(nx),$$

where we recall that $\tau(n)$ is the number of divisors of n, and $\mathrm{e}(z) := \mathrm{e}^{2\pi i z}$. This series converges absolutely for $\mathrm{Re}(s) > 1$ and $x \in \mathbf{R}$ arbitrary. When $x \in \mathbf{Q}$, Estermann (Estermann 1930) proved that the function $D(\cdot, x)$ has a meromorphic continuation to $\mathbf{C}$, and a functional equation relating the value $D(s, a/q)$ to $D(1-s, -d/q)$ when $ad \equiv 1 \pmod{q}$. This can be seen, for instance, by expressing the periodic function $n \mapsto \mathrm{e}(nx)$ in terms of Dirichlet characters.

The value $D(1/2, a/q)$ is then closely linked with the central values $L(\frac{1}{2}, \chi)$ of Dirichlet L functions. Bettin proves in (Bettin 2016) that when q is a prime number and $(a,q) = 1$, the twisted second moment

$$M_2(q,a) := \frac{1}{q^{1/2}} \sum_{\chi \pmod{q}} \chi(a) |L(\tfrac{1}{2}, \chi)|^2$$

is close to the value at $x = a/q$ of the central value of the Estermann function,

$$M_2(q,a) \simeq \mathrm{Re}\, D(\tfrac{1}{2}, a/q) + \mathrm{Im}\, D(\tfrac{1}{2}, a/q). \tag{2.1}$$

As Bettin remarks in (Bettin 2016), the estimation of $M_2(q,a)$ is related to that of the fourth moment of Dirichlet L functions,

$$M_4(q) := \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} |L(\tfrac{1}{2}, \chi)|^4,$$

whose estimation with power-saving error term in q by Young (Young 2011), successively improved in (Blomer ; Fouvry ; Kowalski et al. 2017 ; Wu 2020), was a remarkable success of the methods described in the first part (algebraic exponential sums, Kuznetsov formula). Indeed, we have by orthogonality of characters

$$M_4(q) = \sum_{a \pmod q} |M_2(q,a)|^2.$$

In view of (2.1), this motivates the question of understanding the size of values taken by $D(\frac{1}{2}, x)$ as x varies in a set of rational numbers.

2.1 Statistical distribution of central values of additive twist L functions

2.1.1 Equidistribution of rational orbits for the Gauss map

Improving earlier results by Young (Kıral ; Young 2019) and Conrey (Conrey [s. d.]), Bettin (Bettin 2016) shows that the function

$$\phi(x) := D(\tfrac{1}{2}, x) - D(\tfrac{1}{2}, -\tfrac{1}{x}), \tag{2.2}$$

initially defined for $x \in \mathbf{Q} \setminus \{0\}$, extends to a Hölder continuous function on $\mathbf{R} \setminus \{0\}$. Along with 1-periodicity, this implies more generally that for any $\gamma \in \mathrm{SL}(2, \mathbf{Z})$, the map

$$\phi_\gamma(x) : x \mapsto D(\tfrac{1}{2}, x) - D(\tfrac{1}{2}, \gamma x)$$

extends to a Hölder continuous function on $\mathbf{R} \setminus \{\gamma^{-1}\infty\}$. This shows that $D(\frac{1}{2}, \cdot)$ is an instance of what Zagier called, in a seminal article (Zagier 2010), a "quantum" modular form for $\mathrm{SL}(2, \mathbf{Z})$.

The equality (2.2) paves the way to an expression of $D(\frac{1}{2}, x)$ in terms of the finite orbit of the rational x under the Gauss map

$$T :]0,1[\to [0,1[, \qquad T(x) = \{1/x\},$$

where $\{\cdot\}$ denotes the fractional part of x, that is to say, using the Euclid algorithm :

$$D(\tfrac{1}{2}, x) = D(\tfrac{1}{2}, 0) + \sum_{j=1}^{r(x)} \phi((-1)^{j-1} T^{j-1}(x)), \tag{2.3}$$

where $r(x) \geq 0$ is the smallest integer such that $T^{r(x)}(x) = 0$. This equality brings the problem entirely to that of the distribution on average of orbits of rational numbers under the Gauss map. All the arithmetic in the problem is encoded in the analytic properties of the map ϕ, which is usually called the "observable" in the theory of dynamical systems.

Denote the set of rational numbers $[0, 1[$ of fixed denominator q by

$$\Omega_q := \{a/q, \ 0 \leq a < q, (a, q) = 1\}.$$

The question we would like to be able to address is the following : if x is a rational number taken uniformly at random from Ω_q, what can we say of the random variable $D(\frac{1}{2}, x)$ as $q \to \infty$?

The methods we have at our disposal do not give us any precise results on Ω_q [1]. The problem becomes tractable as soon as we perform an additional average over q, that is to say, if we work in the larger set

$$\Omega_{\leq Q} := \bigcup_{q \leq Q} \Omega_q$$

of rationals of denominators at most Q. The distribution on average of Birkhoff sums of the shape (2.3), among rationals in $\Omega_{\leq Q}$ is a subject which recently entered within the full scope of the theory of measured dynamical system, due to the work of Vallée (Vallée 2000) and Baladi and Vallée (Baladi ; Vallée 2005).

Letting $\Omega_{\leq Q}$ be endowed with the uniform counting measure, we denote $\mathbb{P}_Q$ and $\mathbb{E}_Q$ the associated probability and variance. One of the main results of Baladi and Vallée in (Baladi ; Vallée 2005) is the following.

Théorème 10. *Let $(c(n))_{n \geq 1}$ be a sequence of real numbers satisfying $c(n) = O(\log n)$ for $n \geq 2$. For any $x \in \mathbf{Q} \cap (0, 1)$, we let*

$$f(x) = \sum_{j=1}^{r(x)} c(a_j(x)),$$

where $a_j(x) = \lfloor 1/T^{j-1}(x) \rfloor$ denotes the j-th coefficient in the continued fraction expansion

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}.$$

There exists $\delta > 0$ and two functions U, V defined and holomorphic in a neighborhood $W \subset \mathbf{C}$ of 0, such that as $Q \to \infty$, we have

$$\mathbb{E}_Q\Big(\exp\big(w f(x)\big)\Big) = \exp\big(U(w) \log Q + V(w) + O(Q^{-\delta})\big) \qquad (w \in W). \tag{2.4}$$

The quality of the error term in the right-hand side of (2.4) relies notably on an adapted version of a method of Dolgopyat (Dolgopyat 1998) which gives a uniform spectral gap for the family of transfor operators associated with T,

$$\mathbb{H}_\tau : f \mapsto \sum_{n=1}^{\infty} \frac{1}{(n+x)^{2+i\tau}} f\Big(\frac{1}{n+x}\Big) \qquad (\tau \in \mathbf{R}).$$

As explained in Section 1.1 of (Baladi ; Vallée 2005), analogues of theorem 10

1. We refer nonetheless to (Aistleitner ; Borda ; Hauke 2022) for some quite general recent results.

were known and classical in the situation where $x \in [0,1]$ is chosen uniformly at random according to the Lebesgue measure, and when the function studied is $f_N(x) = \sum_{j=1}^N c(a_j(x))$ with $N \to \infty$. The parameter N plays formally the role of $\log Q$, up to a constant, which is the typical length of the continued fraction expansion of a fraction in $\Omega_{\leq Q}$. The analogous continuous problem is simpler by many aspects : for instance, the continuous analogue of the estimate (2.4) can be obtained using information on perturbations of the single operator $\mathbb{H}_\tau$ for $\tau = 0$ only, which can be gotten using the theory of perturbation of operators, without having to use the techniques of Dolgopyat.

In theorem 10, the function $f(x)$ can be seen as a Birkhoff sum of the same nature as (2.3), where the observable is $\phi(x) := c(\lfloor 1/x \rfloor)$. For the map $D(\frac12, \cdot)$ which we are interested in here, the main difference with respect to the statement of theorem 10 are the following [2] :

1. our observable ϕ is Hölder continuous, rather than constant, separately on each interval $[\frac{1}{n+1}, \frac1n]$,
2. our observable ϕ does not have an exponential moment, more precisely, we have $\phi(x) \sim cx^{-1/2}\log(1/x)$ as $x \to 0$.

In (Bettin ; Drappeau 2022b), we obtain a generalization of theorem 10 which resolves both points. Given a map $\phi : [0,1] \to \mathbf{R}$, denote

$$S_\phi(x) = \sum_{j=1}^{r(x)} \phi(T^{j-1}(x)), \qquad (x \in \mathbf{Q} \cap (0,1)).$$

In the case of theorem 10, we therefore have $f(x) = S_\phi(x)$ for $\phi(x) = c(\lfloor 1/x \rfloor)$. In the case of the Estermann function, we have $D(\frac12, x) = D(\frac12, 0) + S_\phi(x)$ where ϕ is a certain Hölder continuous map.

The statement of our result involves the integral

$$I_\phi(t) := \int_0^1 (\mathrm{e}^{it\phi(x)} - 1) \frac{\mathrm{d}x}{(1+x)\log 2} \qquad (t \in \mathbf{R}). \tag{2.5}$$

Théorème 11 (Bettin ; Drappeau 2022b)**.** *Let $\kappa_0, \lambda_0, \alpha_0 > 0$. Assume that the function $\phi : (0,1) \to \mathbf{R}$ satisfies the following properties :*

- *For each $n \in \mathbf{N}$, the map $\phi|_{]1/(n+1),1/n[}$ extends to a κ_0-Hölder continuous map,*
- *We have*

$$\sum_{n \geq 1} \frac{1}{n^2} \Big(\sup_{x \in [\frac{1}{n+1}, \frac1n]} |\phi(x)|^{\alpha_0} + \sup_{x,y \in [\frac{1}{n+1}, \frac1n]} \frac{|\phi(x) - \phi(y)|^{\lambda_0}}{|x-y|^{\lambda_0 \kappa_0}} \Big) < \infty. \tag{2.6}$$

2. We neglect in what follows the effect of the sign change $(-1)^{j-1}$ in (2.3).

Then there exists $t_0, \delta > 0$ and two maps $U, V : [-t_0, t_0] \to \mathbf{C}$ such that

$$\mathbb{E}_Q\Big(\exp\big(itS_\phi(x)\big)\Big) = \exp\big(U(t)\log Q + V(t) + O(Q^{-\delta})\big), \qquad (t \in [-t_0, t_0]), \tag{2.7}$$

as well as

$$U(t) = \frac{12\log 2}{\pi^2} I_\phi(t) + O_\varepsilon(t^2 + |t|^{2\alpha_0 - \varepsilon}),$$
$$V(t) = O_\varepsilon(|t| + |t|^{\alpha_0 - \varepsilon}).$$

If moreover we have $\alpha_0 > 1$, then there exists $C \in \mathbf{R}$ such that

$$U(t) = \frac{12\log 2}{\pi^2} I_\phi(t) + Ct^2 + O_\varepsilon(t^3 + |t|^{1+\alpha_0-\varepsilon}).$$

The conclusion of theorem 11 corresponds to the case $t = iw$ of theorem 10 of Baladi-Vallée. An estimate for t complex in the neighborhood of 0 can be obtained, with an identical proof, under the natural condition that ϕ admits exponential moments, in the sense that

$$\sum_{n\geq 1} \frac{1}{n^2} \sup_{x\in[\frac{1}{n+1},\frac{1}{n}]} \exp\big(\sigma_0|\phi(x)|\big) < \infty$$

for some $\sigma_0 > 0$. This is equivalent to asking that $\phi(x) = O(|\log(2/x)|)$, similarly as in the growth hypothesis on ϕ in theorem 10. This hypothesis is not verified in the case of the Estermann function.

The proof that we present with Bettin of theorem 11 relies on a modification of the arguments of Baladi and Vallée (Baladi ; Vallée 2005), and particularly the part which uses the works of Dolgopyat (Dolgopyat 1998). To resolve the first objection 1., we work in a space of Hölder continuous functions, which can be done using standard analytical methods. This modification turned out to be very useful to allow arbitrary large values of $\lambda_0 > 0$ in condition (2.6). To resolve the second objection 2., we use recent results of Kloeckner (Kloeckner 2019) on the spectrum of perturbed linear operators, which are valid without having to assume analyticity of the perturbation. The price to pay to obtain weak hypotheses on ϕ is to express our results in terms of the integral $I_\phi(t)$ without further evaluating it.

In a companion paper (Bettin ; Drappeau 2022a), we evaluate the integral $I_\phi(t)$ in most practical situations. In the most favourable cases, ϕ is square integrable, and we can estimate the integral (2.5) by inserting a Taylar expansion of order 2. We obtain the following corollary.

Corollaire 3 (Bettin ; Drappeau 2022b). *Assume the hypotheses and notations of theorem 11. Assume moreover that $\alpha_0 \geq 2$, and that there does not exist a constant $c \in \mathbf{R}$ and a map f on $[0, 1]$ such that $\phi = c + f - f \circ T$. Then there* *exists $\mu \in \mathbf{R}$ and $\sigma > 0$ such that for all $t \in \mathbf{R}$, as $Q \to \infty$, we have*

$$\mathbb{P}_Q\Big(\frac{S_\phi(x) - \mu \log Q}{\sigma\sqrt{\log Q}} \leq t\Big) = \int_{-\infty}^{t} \frac{\mathrm{e}^{-v^2/2}\,\mathrm{d}v}{\sqrt{2\pi}} + o(1).$$

The expression of μ here is

$$\mu = \frac{12}{\pi^2}\int_0^1 \phi(x)\frac{\mathrm{d}x}{1+x}.$$

The quantity σ, on the other hand, does not seem to admit in general a simple explicit expression in terms of ϕ.

Lastly, we illustrate theorem 11 by a limit theorem on the sum of coefficients $a_j(x)$ of the continued fraction expansion,

$$\Sigma(x) := \sum_{j=1}^{r(x)} a_j(x).$$

We obviously have $\Sigma(x) = S_\phi(x)$ for the choice $\phi(x) = \lfloor 1/x \rfloor$. This map ϕ is not integrable near 0. To state the limit law, we define G_1 as the cumulative distribution function of the stable law $S_1(\frac{6}{\pi}, 1, 0)$, which is to say,

$$G_1(v) := \int_{-\infty}^{v} g_1(x)\,\mathrm{d}x, \qquad g_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-itx}\,\mathrm{e}^{-\frac{6}{\pi}|t| - \frac{12}{\pi^2}it\log|t|}\,\mathrm{d}t. \tag{2.8}$$

We also denote γ_0 the Euler-Mascheroni constant.

Corollaire 4 (Bettin ; Drappeau 2022b)**.** *For each $v \in \mathbf{R}$, we have*

$$\mathbb{P}_Q\Big(\frac{\Sigma(x)}{\log Q} - \frac{\log\log Q - \gamma_0}{\pi^2/12} \leq v\Big) \to G_1(v)$$

as $Q \to \infty$.

The law $S_1(\frac{6}{\pi}, 1, 0)$ does not a have a moment of order 1. This corresponds to the fact that $\phi(x) = \lfloor 1/x \rfloor$ is not integrable on $[0, 1]$.

The continuous analogue of corollary 4, that is, when x is taken uniformly at random from $[0, 1]$ (the sum $\Sigma(x)$ being truncated at $j \leq N$ for a parameter $N \in \mathbf{N}$ which tends to infinity), was established by Heinrich (Heinrich 1987).

2.1.2 Application to the Estermann function

We can now apply theorem 11 to the map $D(\frac{1}{2}, \cdot)$. To simplify the exposition, we consider its real part, which has the advantage of being an even function, and we let $\psi = \operatorname{Re}\phi$ where ϕ is defined by (2.2). As Bettin showed in (Bettin 2016), the map ψ essentially takes the shape

$$\psi(x) = x^{-1/2}(c_1 \log x + c_2) + E(x),$$

where E is Hölder continuous on $\mathbf{R}$ of exponent α for all $\alpha < 1/2$, and c_1, c_2 are constants. In (Bettin ; Drappeau 2022a), the integral $I_\psi(t)$ is estimated by

$$I_\psi(t) = 1 + i\mu t - \sigma^2 t^2 |\log t|^3 + o(t^2 |\log t|^3)$$

as $t \to 0^+$, where $\mu \in \mathbf{R}$ is a constant and $\sigma = 1/\pi$. By estimating the expectation of $D(1/2, x)$ on $\Omega_{\leq Q}$, we show that $\mu = 0$. We then deduce the following result, which answers our initial question.

Corollaire 5 (Bettin ; Drappeau 2022b)**.** *For $\sigma = 1/\pi$, we have*

$$\mathbb{P}_Q\Big(\frac{\operatorname{Re} D(x)}{\sigma\sqrt{(\log Q)(\log\log Q)^3}} \leq t\Big) = \int_{-\infty}^{t} \frac{\mathrm{e}^{-v^2/2}\,\mathrm{d}v}{\sqrt{2\pi}} + o(1)$$

as $Q \to \infty$.

The appearance of the additional factor $(\log\log Q)^3$, which we did not anticipate, is intimately linked with the fact that the map ψ is not square integrable. This follows *in fine* from the presence of a double pole at $s = 1$ for the Dirichlet series $\zeta(s)^2$, of which $D(s, x)$ is the additive twist.

The simplicity with which corollary 5 is deduced from theorem 11 obfuscates our many attempts to obtain corollary 5 by other methods, which have all failed. As an example, the most natural strategy was to pass through an estimation of moments, in order to take advantage of orthogonality of additive characters. This computation was performed by Bettin in (Bettin 2019) with the conclusion that the estimation is no longer relevant starting from the second moment, due to a dominant contribution coming from an asymptotically negligible set of rational numbers. To undertake this strategy would have required a fine understanding of the rationals which cause this phenomenon, and an analytically tractable way of taking away their contribution. On the contrary, the dynamical methods which underly theorem 11 analyze this obstacle in a natural and transparent way.

About the link with moments of Dirichlet L-functions, which we discussed at the beginning of this section, it is disappointing that the exact relation (2.1), between the twisted second moment of Dirichlet L-functions and the Estermann function, only holds when q is prime ; when q is not prime, this relation is perturbed by an additional multiplicative convolution. This does not allow us to transfer immediately the above results, which are valid on average over q without constraints. We can nonetheless make a plausible conjecture on the distribution of values of the twisted moments $M_2(q, a)$.

Conjecture 2. *The multiset*

$$\left\{\frac{M_2(a,q)}{(\log q)^{1/2}(\log\log q)^{3/2}},\ 0<a<q,(a,q)=1\right\}$$

tends to distribute, as $q\to\infty$, according to a centered Gaussian law.

2.1.3 Application to central values of additive twists of modular forms L-functions

Theorem 11 found an application to another type of maps on $\mathbf{Q}$, which are built out of modular forms. Let f be a cuspidal holomorphic modular form of weight k and level N, which we denote $f\in S_k(N)$. Let

$$f(z)=\sum_{n\geq 1}a_n n^{(k-1)/2}\,\mathrm{e}(nz),\qquad (\mathrm{Im}(z)>0)$$

be the Fourier expansion of f at infinity. We define the L-function

$$L(f,s)=\sum_{n\geq 1}\frac{a_n}{n^s},$$

initially on $\mathrm{Re}(s)>1$. It is known since Hecke that this map extends to an entire function of s, which has a functional equation of the shape

$$L(f,s)=\gamma_f(s)L(\bar{f},1-s)$$

where γ_f is essentially a quotient of values of Euler's Γ function. When f is a Hecke eigenform, the map $n\mapsto a_n$ is multiplicative, and $L(f,s)$ then possesses an Euler product. Hecke's method consists in expressing $L(f,s)$ as an integral, or period, of the form f :

$$L(f,s)=\frac{(2\pi)^s}{i\Gamma(s)}\int_0^{i\infty}f(z)(\mathrm{Im}\,z)^{s+\frac{k-1}{2}-1}\,\mathrm{d}z,\qquad (s\in\mathbf{C}),\tag{2.9}$$

where the integration path is a vertical line.

Analogously to the construction of the Estermann function, we can form the additive twist :

$$L(f,s,x):=\sum_{n\geq 1}\frac{a_n}{n^s}\,\mathrm{e}(nx)$$

which converges for $\mathrm{Re}(s)>1$ and $x\in\mathbf{R}$. We have an expression analogous to that of Hecke (2.9) :

$$L(f,s,x)=\frac{(2\pi)^s}{i\Gamma(s)}\int_x^{i\infty}f(z)(\mathrm{Im}\,z)^{s+\frac{k-1}{2}-1}\,\mathrm{d}z,\qquad (\mathrm{Re}(s)>1),$$

the integration path being again a vertical line. As $x \in \mathbf{Q}$, the decay of f near x along the half-line $\{x + iy,\ y > 0\}$ allows to analytically extend the integral on the right-hand side, and thus also the map $L(f, \cdot, x)$, to the whole complex plane. The resulting function again satisfies a functional equation of the shape $L(f, s, a/q) \leftrightarrow L(f, 1-s, -d/q)$ when $ad \equiv 1 \pmod q$. We can define the central value

$$L_f(x) := L(f, 1/2, x) \qquad (x \in \mathbf{Q}).$$

When $k = 2$ we find, up to a multiplicative factor, a quantity known as the modular symbol

$$L_f(x) = c_f \langle x \rangle_f, \qquad \langle x \rangle_f := \int_x^{i\infty} f(z)\,\mathrm{d}z \qquad (k = 2).$$

Modular symbols are used, in particular, to compute explicitely modular forms (Cremona 1997).

Motivated by applications related to the rank of the group of rational points of an elliptic curve along a family of extensions, Mazur and Rubin formulated a series of conjectures on the distribution of values of modular symbols $\langle x \rangle_f$ on average over x (Mazur; Rubin 2021). Their original conjectures were made for x with given fixed denominator, that is, for $x \in \Omega_q$ with the notations of the previous section. These conjectures are still open and presumed to be very difficult. The computation of the second moment of $\langle x \rangle_f$ lies very close to the threshold of known techniques (Blomer; Fouvry; Kowalski et al. 2023).

Still when $k = 2$, Petridis and Risager (Petridis; Risager 2018) showed that the question could be resolved by performing an additional average over denominators, which means picking $x \in \Omega_{\leq Q}$ with the notations of the previous section. Their main theorem applies to any given holomorphic modular form of weight 2 on a Fuchsian group of the first kind, and implies in particular, for a form $f \in S_k(N)$, the existence of a constant $\sigma_f > 0$ such that for all $t \in \mathbf{R}$,

$$\mathbb{P}_Q\Big(\frac{L_f(x)}{\sigma_f \sqrt{\log Q}} \leq t\Big) \to \int_{-\infty}^t \frac{\mathrm{e}^{-v^2/2}\,\mathrm{d}v}{\sqrt{2\pi}}, \qquad (Q \to \infty). \tag{2.10}$$

The method of Petridis and Risager (Petridis; Risager 2018) uses the computation of a moment generating function, and is capable of giving a statement of type “quasi-powers” analogous to (2.4).

When $k = 2$, one of the known properties of the modular symbol translates into the fact that the map

$$x \mapsto L_f(x) - L_f(\gamma x)$$

is constant for any fixed $\gamma \in \Gamma_0(N)$ (*cf.* (Mazur; Rubin 2021)). This property is crucial in the setup of (Petridis; Risager 2018). This certainly shows that $x \mapsto L_f(x)$ is a quantum modular form in the sense of Zagier.

Consider now the case when the weight $k \in \mathbf{Z}$ is larger than 2. For all $\gamma \in \Gamma_0(N)$, let

$$\phi_\gamma(x) : x \mapsto L_f(x) - L_f(\gamma x) \tag{2.11}$$

When $k > 2$, the map $x \mapsto L_f(x) - L_f(\gamma x)$ for $\gamma \in \Gamma_0(N)$ is no longer necessarily constant. However, as it is proven in (Bettin ; Drappeau 2022b ; Nordentoft 2021a), all of the maps ϕ_γ extend to functions on $\mathbf{R}$ which are $(1-\varepsilon)$-Hölder continuous and bounded in $x \in \mathbf{R}$. When $N = 1$, that is when f is modular with respect to $\mathrm{SL}(2, \mathbf{Z})$, corollary 3 allows us to deduce the following central limit theorem on the values $L_f(x)$ for $x \in \Omega_{\leq Q}$, which is an analogue of the theorem of Petridis-Risager where the hypothesis "$f \in S_2(N)$" is replaced by "$f \in S_k(1)$".

Théorème 12 (Bettin ; Drappeau 2022b). *Assume that $f \in S_k(1)$. There exists $\sigma_f > 0$ such that the estimate (2.10) holds for all $t \in \mathbf{R}$, as $Q \to \infty$.*

The new aspect of our work is the fact that we use no information on f besides a weak hypothesis on the regularity of ϕ_γ. It is the strategy of the method, based on the expression in terms of iterates of the Gauss map, which allows this.

Lee and Sun (Lee ; Sun 2019), independently of us and roughly at the same time, had the same idea of using Baladi-Vallée's strategy on this problem, and could handle the case when $f \in S_2(N)$. A main novel ingredient in their work is the idea to use a natural extension of the Gauss map to handle congruence groups.

At about the same time, Nordentoft (Nordentoft 2021b) obtained the estimate of theorem 12 for any holomorphic modular form f on a Fuchsian group of the first kind, which includes theorem 12 as a special case. Nordentoft's method is different form Petridis-Risager's, Lee-Sun's and ours, but is more related to Petridis-Risager's, and goes through the computation of moments.

In a recent work with Nordentoft (Drappeau ; Nordentoft 2022), we obtained a result on the regularity on the maps (2.11) in the general case of a Maaß form on a Fuchsian subgroup of $\mathrm{SL}(2, \mathbf{R})$ of the first kind, which generalizes both the cases of holomorphic forms (Nordentoft's theorem) which we just discussed, and also the situation of the Estermann function $a_n = \tau(n)$, which corresponds to a special case when the Maaß form is a certain Eisenstein series. This allows us to prove a central limit theorem analogous to theorem 12 when f is replaced by a Maaß form for the particular group $\mathrm{SL}(2, \mathbf{Z})$.

In a work in progress with Bettin and Lee, we are trying to work out an analogue of theorem 11 for the action of a general Fuchsian group of the first kind on the set of its cusps. When this work will have converged, we hope to have a limit law similar to (2.10) valid for all twisted L-values of L-functions of rank 2, in the sense of (Iwaniec ; Kowalski 2004, Chapitre V).

2.2 Value distribution of a quantum knot invariant

In Zagier's article (Zagier 2010) where he introduces quantum modular forms, a singular example arises from Kashaev invariants in knot theory.. The exact definition of this invariant in general involves a very delicate combinatorial construction starting from a knot diagram. In the simplest non-trivial case, that of the "figure eight" knot denotes 4_1, this invariant takes the shape of a function $J : \mathbf{Q} \to \mathbf{R}$ given by

$$J(x) = \sum_{n=0}^{\infty} \prod_{r=1}^{n} |1 - \mathrm{e}(rx)|^2. \tag{2.12}$$

The sum is finite when $x \in \mathbf{Q}$. Zagier noticed numerically that the map h defined by

$$\log J(x) - \log J(1/x) = h(x) \tag{2.13}$$

seems to extend to $\mathbf{R} \setminus \{0\}$ as a somewhat regular function, for instance continuous almost everywhere, whereas the function J itself is very irregular. These two maps are drawn on figures 3 and 4 of (Zagier 2010).

Conjecture 3 (Zagier). — *The map h extends to a function on $\mathbf{R} \setminus \{0\}$ which is continuous at irrationals, and left- and right-continuous at rationals.*
— *We have $h(x) = \frac{C}{x} - \frac{3}{2} \log x + O(1)$ for $x \in]0, 1]$, where $C = \mathrm{Vol}(4_1)/2\pi \approx 0,323$ and $\mathrm{Vol}(4_1)$ is the hyperbolic volume of the figure eight knot complement.*

This map gives the opportunity of an interesting application of theorem 11. Indeed, by definition of h and using the value $J(0) = 1$, we have

$$\log J(x) = \sum_{j=1}^{r(x)} h(T^{j-1}(x)). \tag{2.14}$$

The function h does not exactly satisfy the regularity hypotheses stated in theorem 11, but it is not very far from it. If conjecture 3 was true, we would deduce the convergence in law to a stable law of parameter 1 similar to (2.8), for the values taken by $\log J(x)$ at rationals ordered by denominators.

We therefore went looking with Sandro Bettin for a proof of Zagier's conjecture. This led us to study more precisely the q-Pochhammer symbol,

$$(q)_N := (1 - q) \dots (1 - q^N), \qquad (N \in \mathbf{N}), \tag{2.15}$$

whose modulus squared is involved in the definition (2.12). This sort of product is also called, in other contexts, a Sudler product (Sudler 1964), and we will come back to this later on.

The symbol $(q)_N$ can be viewed as a truncated version of the Dedekind η-function,

$$\eta(z)=\prod_{n\geq 1}(1-\mathrm{e}(nz)),\qquad (\mathrm{Im}(z)>0),$$

which is a modular form of weight $1/2$.

One of the two main results in the work (Bettin ; Drappeau 2022c) with Sandro Bettin is a form of modularity for the q-Pochhammer symbol. We denote $\mathrm{den}(x)$ the denominator of $x\in\mathbf{Q}$. Stated in an informal way, our result can be expressed as follows.

Théorème 13 (Bettin ; Drappeau 2022c)**.** *Let $\gamma\in\mathrm{SL}(2,\mathbf{Z})$. For $1\leq r<\mathrm{den}(\gamma x)$, we have*

$$\mathrm{e}(\tfrac{\gamma x}{24})(\mathrm{e}(\gamma x))_r=\chi(\gamma)\,\mathrm{e}(\tfrac{x}{24})(\mathrm{e}(x))_{r'}\psi_\gamma(x,r),\tag{2.16}$$

for some $1\leq r'<\mathrm{den}(x)$, where χ is the central character ("nebentypus") associated to the Dedekind function (Iwaniec 1997, p. 45), and ψ_γ is a map satisfying certain holomorphicity conditions.

The variable with respect to which ψ_γ extends as a holomorphic function is $z=\{r\,\mathrm{den}(x)/\,\mathrm{den}(\gamma x)\}$. Our starting point is the summation formula of Abel-Plana applied to the sum

$$\sum_{n=1}^{N}\log(1-\mathrm{e}(nz)).$$

The Abel-Plana formula is a version of the classical Euler-Maclaurin formula, where the error term is expressed in an exact way as an integral of the summand, here $\log(1-\mathrm{e}(z))$. Our argument exploits, implicitely, the fact that the function $\log(1-\mathrm{e}(z))$ is close to a primitive of the function $\pi\cot(\pi z)$, which is the summation kernel associated to integers (it has a simple pole of residue 1 at every integer).

The fact that formula (2.16) is exact, as well as the holomorphicity property we mentioned about ψ_r, are two crucial aspects when one tries to sum this formula over r. Here we strongly relied on work of Ohtsuki Ohtsuki 2016. This has allowed us to obtain a proof of Zagier's conjecture for certain special cases of hyperbolic knots [3]. In the case of the figure eight knot, we deduce the following property.

Théorème 14 (Bettin ; Drappeau 2022c)**.** *There exists a sequence (c_n) of real numbers such that for all $Q,N\geq 1$ and $x\in\,]0,1]$, if $\mathrm{den}(x)\leq Q$, we have*

$$h(x)=\frac{C}{x}-\frac{3}{2}\log x+\sum_{n=0}^{N-1}c_nx^n+O_{Q,N}(x^N).$$

3. The few knots we study are essentially those for which the *volume conjecture* of Kashaev (Kashaev 1997) is known ; conversely, Zagier's conjectures implies the volume conjecture.

Now equipped with such a property, we would like to be able to deduce, through the definition (2.13), a limiting law for the values $\log J(x)$. Unfortunately, the fact that the error term depends implicitely on the denominator of x forbids us to undergo this strategy at this point. We found an indirect successful path, however, by complementing theorem 13 with another reciprocity formula.

Write $x = [0; a_1, \dots, a_r]$ the continued fraction expansion of x. Applying the Gauss map consists in taking away from x its first coefficient,

$$T(x) = [0; a_2, \dots, a_r].$$

In certain circumstances (Bettin ; Conrey 2013), it is analytically beneficial to try to take away its last coefficient instead,

$$U(x) := [0; a_1, \dots, a_{r-1}].$$

This amounts, up to a sign, to conjugating the Gauss map by the involution which associates to a fraction $a/q \in \,]0,1[$ the fraction $(-1)^{r-1}\bar{a}/q \pmod 1$, where $a\bar{a} \equiv 1 \pmod q$. Indeed this involution has the effect of reversing the continued fraction expansion.

The Bezout relation

$$\frac{\bar{a}}{q} + \frac{\bar{q}}{a} = \frac{1}{aq} \pmod 1$$

is a precise version of the assertion that the numbers x and $U(x)$ are close to each other, which can also be seen from the definition of U.

We obtain the following relation.

Théorème 15 (Bettin ; Drappeau 2022c). *We have*

$$\log J(x) = \log J(U(x)) + Ca_r(x) + E(x),$$

where $E(x)$ is an error term bounded by $\log a_r(x)$ in most circumstances.

This theorem is specific to the knot 4_1, contrarily to theorems 13 and 14. Treating the error term required a specific technical lemma about partial sums of the cotangent function (Bettin ; Drappeau 2020), which was one of the most troublesome part of the argument.

The conjunction of theorems 13 and 15 yields just enough information on the function J to obtain, through the iteration (2.14), the following approximation. We recall that $\Sigma(x) = \sum_{j=1}^{r(x)} a_j(x)$.

Théorème 16 (Bettin ; Drappeau 2022c). *For all $x \in \mathbf{Q} \cap \,]0,1[$, we have*

$$\log J(x) \sim C\Sigma(x), \qquad \frac{\Sigma(x)}{r(x)} \to \infty.$$

On the one hand, it is classically known that $r(x) \ll \log Q$ when $\mathrm{den}(x) \leq Q$. On the other hand, corollary 4 ensures that

$$\Sigma(x) \sim \frac{12}{\pi^2}(\log Q)\log\log Q$$

for a proportion $1+o(1)$ of rationals $x \in \Omega_Q$ [4]. We immediately deduce the following corollary, which can be seen as a law of large numbers for the values of $\log J(x)$ for $x \in \Omega_Q$.

Corollaire 6 (Bettin ; Drappeau 2022c). *There exists a map $\varepsilon : \mathbf{N} \to \mathbf{R}_+$ with $\varepsilon(Q) \to 0$ as $Q \to \infty$, such that*

$$\log J(x) = \Big(\frac{12}{\pi^2}C + O(\varepsilon(Q))\Big)(\log Q)\log\log Q$$

for all $x \in \Omega_Q$, with at most $\varepsilon(Q)Q^2$ exceptions.

Very recently, Aistleitner and Borda obtained several results on $J(x)$ using their works on perturbations of products of the sine function (Borda ; Aistleitner 2022c ; Borda ; Aistleitner 2022b ; Borda ; Aistleitner 2022a). In particular, it is proved in corollary 2 of (Borda ; Aistleitner 2022a) that the map h defined in (2.13) can be extended by density to a function which is almost everywhere continuous on $\mathbf{R} \setminus \{0\}$. More precisely, Aistleitner and Borda show that this extension of the function h is continuous at those irrationals x which are "well-approximable" in the sense that the sequence $(a_j(x))_j$ is unbounded. This allowed them to obtain, in theorem 4 of (Borda ; Aistleitner 2022a), a central limit theorem for $\log J(x)$, of the same precision as corollary 4.

One of the main questions left open is the continuity of h at badly approximable rationals, in particular at quadratic irrationals such as $1/\phi = [0; 1, 1, \dots]$. This is the topic of a work in progress with Sandro Bettin and Bence Borda. We have good hope in the case of quadratic irrationals.

Another lead we wish to explore, but on which we have no serious angle of attack, is to the validity of the law of large numbers of the same type as corollary 6, for more general knots. Our proof for the knot 4_1 uses crucially the positivity of the summands in (2.12), and a new idea is required in order to dispense with this ingredient. It is more precisely our proof of theorem 15 which does not seem to be well-suited to other cases. The works of Aistleitner-Borda (Borda ; Aistleitner 2022a), as well as our work in progress with Sandro Bettin and Bence Borda on the function h, may give another starting point.

4. Actually this is true for a proportion $1+o(1)$ of rationals $x \in \Omega_q$, without averaging over the denominators, thanks to works of Rukavishnikova (Rukavishnikova 2011) ; see also the recent paper (Aistleitner ; Borda ; Hauke 2022).

Bibliographie

AISTLEITNER, C. ; BORDA, B. ; HAUKE, M., 2022. *On the distribution of partial quotients of reduced fractions with fixed denominator*. Disp. à l'adr. eprint : http://arxiv.org/abs/2210.14095. Prépublication (cf. p. 29, 40).

ASSING, E. ; BLOMER, V. ; LI, J., 2021. Uniform Titchmarsh divisor problems. *Adv. Math.* T. 393, p. 51. Id/No 108076 (cf. p. 10, 11).

BALADI, V. ; VALLÉE, B., 2005. Euclidean algorithms are Gaussian. *J. Number Theory*. T. 110, nº 2, p. 331-386 (cf. p. 29, 31).

BETTIN, S., 2016. On the reciprocity law for the twisted second moment of Dirichlet L-functions. *Trans. Amer. Math. Soc.* T. 368, nº 10, p. 6887-6914 (cf. p. 27, 28, 32).

BETTIN, S., 2019. High moments of the Estermann function. *Algebra Number Theory*. T. 13, nº 2, p. 251-300 (cf. p. 33).

BETTIN, S. ; BUI, H. ; LI, X. et al., 2020. A quadratic divisor problem and moments of the Riemann zeta-function. *J. Eur. Math. Soc. (JEMS)*. T. 22, nº 12, p. 3953-3980 (cf. p. 10).

BETTIN, S. ; CONREY, B., 2013. Period functions and cotangent sums. *Algebra Number Theory*. T. 7, nº 1, p. 215-242 (cf. p. 39).

BETTIN, S. ; DRAPPEAU, S., 2020. Partial sums of the cotangent function. *J. Théor. Nombres Bordeaux*. T. 32, nº 1, p. 217-230 (cf. p. 39).

BETTIN, S. ; DRAPPEAU, S., 2022a. Effective estimation of some oscillatory integrals related to infinitely divisible distributions. *Ramanujan J.* T. 57, nº 2, p. 849-861 (cf. p. 31, 33).

BETTIN, S. ; DRAPPEAU, S., 2022b. *Limit laws for rational continued fractions and value distribution of quantum modular forms*. À paraître à Proc. London Math. Soc. (cf. p. 30-33, 36).

BETTIN, S. ; DRAPPEAU, S., 2022c. Modularity and value distribution of quantum invariants of hyperbolic knot. *Math. Ann.* T. 382, nº 3–4, p. 1631-1679 (cf. p. 38-40).

BLOMER, V. ; FOUVRY, É ; KOWALSKI, E. et al., 2017. On moments of twisted *L*-functions. *Amer. J. Math.* T. 139, nº 3, p. 707-768 (cf. p. 27).

BLOMER, V. ; FOUVRY, É. ; KOWALSKI, E. et al., 2023. *The second moment theory of families of L-functions*. Mem. Amer. Math. Soc. Nº 1394 (cf. p. 35).

BLOMER, V. ; MILIČEVIĆ, D., 2015. Kloosterman sums in residue classes. *J. Eur. Math. Soc. (JEMS)*. T. 17, nº 1, p. 51-69 (cf. p. 12).

BOMBIERI, E. ; FRIEDLANDER, J. B. ; IWANIEC, H., 1986. Primes in arithmetic progressions to large moduli. *Acta Math.* T. 156, nº 1, p. 203-251 (cf. p. 10, 15, 16, 22).

BORDA, B. ; AISTLEITNER, C., 2022a. *A conjecture of Zagier and the value distribution of quantum modular forms*. Prépublication (cf. p. 40).

BORDA, B. ; AISTLEITNER, C., 2022b. *Maximizing Sudler products via Ostrowski expansions and cotangent sums*. À paraître à Algebra Number Theory (cf. p. 40).

BORDA, B. ; AISTLEITNER, C., 2022c. Quantum invariants of hyperbolic knots and extreme values of trigonometric products. *Math. Z.* T. 302, p. 759-782 (cf. p. 40).

BUTTCANE, J., 2022. *On sums of hyper-Kloosterman sums*. Disp. à l'adr. eprint : `http://arxiv.org/abs/2205.14194`. Prépublication (cf. p. 12).

CONREY, J. B., [s. d.]. *The mean-square of Dirichlet L-functions*. Disp. à l'adr. eprint : `http://arxiv.org/abs/0708.2699`. AIM preprint number 2007-54 (cf. p. 28).

CREMONA, J. E., 1997. *Algorithms for modular elliptic curves*. Cambridge : Cambridge University Press (cf. p. 35).

DESHOUILLERS, J.-M. ; IWANIEC, H., 1982a. Kloosterman sums and Fourier coefficients of cusp forms. *Invent. Math.* T. 70, nº 2, p. 219-288 (cf. p. 10-12, 16, 23).

DESHOUILLERS, J.-M. ; IWANIEC, H., 1982b. On the greatest prime factor of $n^2 + 1$. *Ann. Inst. Fourier (Grenoble)*. T. 32, nº 4, p. 1-11 (cf. p. 24).

DOLGOPYAT, D., 1998. On Decay of Correlations in Anosov Flows. *Ann. of Math. (2)*. T. 147, nº 2, p. 357-390 (cf. p. 29, 31).

DRAPPEAU, S., 2015. Théorèmes de type Fouvry–Iwaniec pour les entiers friables. *Compos. Math.* T. 151, nº 5, p. 828-862 (cf. p. 20).

DRAPPEAU, S., 2017. Sums of Kloosterman sums in arithmetic progressions, and the error term in the dispersion method. *Proc. London Math. Soc.* T. 114, p. 684-732 (cf. p. 11, 15).

DRAPPEAU, S. ; MAYNARD, J., 2019. Sign changes of Kloosterman sums and exceptional characters. *Proc. Am. Math. Soc.* T. 147, nº 1, p. 61-75 (cf. p. 13).

DRAPPEAU, S. ; NORDENTOFT, A. C., 2022. *Central values of additive twists of Maaß forms L-functions*. Disp. à l'adr. eprint : `http://arxiv.org/abs/2208.14346`. Prépublication (cf. p. 36).

DRAPPEAU, S. ; PRATT, K. ; RADZIWIŁŁ, M., 2022. *One-level density estimates for Dirichlet L-functions with extended support*. À paraître à Algebra Number Theory (cf. p. 11, 21).

DRAPPEAU, S. ; TOPACOGULLARI, B., 2019. Combinatorial identities and Titchmarsh's divisor problem for multiplicative functions. *Algebra Number Theory*. T. 13, nº 10, p. 2383-2425 (cf. p. 16, 19, 20).

DUKE, W. ; FRIEDLANDER, J. B. ; IWANIEC, H., 1994. A quadratic divisor problem. *Invent. Math.* T. 115, nº 1, p. 209-217 (cf. p. 14).

DUKE, W. ; FRIEDLANDER, J. B. ; IWANIEC, H., 1995. Equidistribution of Roots of a Quadratic Congruence to Prime Moduli. *Ann. of Math. (2)*. T. 141, nº 2, p. 423-441 (cf. p. 26).

ESTERMANN, T., 1930. On the Representations of a Number as the Sum of Two Product. *J. London Math. Soc.* T. s1-5, nº 2, p. 131-137 (cf. p. 27).

FOUVRY, É., 1985. Sur le problème des diviseurs de Titchmarsh. *J. Reine Angew. Math.* T. 357, p. 51-76 (cf. p. 15, 16, 22).

FOUVRY, É. ; MICHEL, Ph., 2003. Sommes de modules de sommes d'exponentielles. *Pacific J. Math.* T. 209, nº 2, p. 261-288 (cf. p. 14).

FOUVRY, É. ; MICHEL, Ph., 2007. On the sign change of Kloosterman sums. *Ann. of Math. (2)*. T. 165, nº 3, p. 675-715 (cf. p. 12).

FOUVRY, É. ; TENENBAUM, G., 1990. Diviseurs de Titchmarsh des entiers sans grand facteur premier. In : *Analytic number theory (Tokyo, 1988)*. Springer. T. 1434, p. 86-102. Lecture Notes in Math. (Cf. p. 20).

FOUVRY, É. ; TENENBAUM, G., 2022. Multiplicative functions in large arithmetic progressions and applications. *Trans. Amer. Math. Soc.* T. 375, nº 1, p. 245-299 (cf. p. 16).

FRIEDLANDER, J. B. ; IWANIEC, H., 2003. Exceptional characters and prime numbers in arithmetic progressions. *Int. Math. Res. Not.* T. 2003, nº 37, p. 2033-2050 (cf. p. 13).

FRIEDLANDER, J. B. ; IWANIEC, H., 2004. Exceptional characters and prime numbers in short intervals. *Selecta Math. (N.S.)* T. 10, nº 1, p. 61-69 (cf. p. 13, 14).

FRIEDLANDER, J. B. ; IWANIEC, H., 2005. The illusory sieve. *Int. J. Number Theory*. T. 01, nº 04, p. 459-494 (cf. p. 13, 14).

FRIEDLANDER, J. B. ; IWANIEC, H., 2013. Exceptional discriminants are the sum of a square and a prime. *Q. J. Math.* T. 64, nº 4, p. 1099-1107 (cf. p. 13).

GRANVILLE, A., 2008. Smooth numbers : computational number theory and beyond. In : *Algorithmic number theory : lattices, number fields, curves and cryptography*. Cambridge Univ. Press. T. 44, p. 267-323. Math. Sci. Res. Inst. Publ. (Cf. p. 25).

HARPER, A. J., 2012. *Bombieri-Vinogradov and Barban-Davenport-Halberstam type theorems for smooth numbers*. Disp. à l'adr. eprint : `http://arxiv.org/abs/1208.5992`. Prépublication (cf. p. 20).

HEATH-BROWN, D. R., 1982. Prime numbers in short intervals and a generalized Vaughan identity. *Canad. J. Math.* T. 34, nº 6, p. 1365-1377 (cf. p. 17).

HEATH-BROWN, D. R., 1983. Prime Twins and Siegel Zeros. *Proc. London Math. Soc. (3).* T. s3-47, nº 2, p. 193-224 (cf. p. 13, 14).

HEINRICH, L., 1987. Rates of convergence in stable limit theorems for sums of exponentially ψ-mixing random variables with an application to metric theory of continued fractions. *Math. Nachr.* T. 131, p. 149-165 (cf. p. 32).

HILDEBRAND, A. ; TENENBAUM, G., 1993. Integers without large prime factors. *J. Théor. Nombres Bordeaux.* T. 5, p. 411-484 (cf. p. 25).

HMYROVA, N. A., 1964. On polynomials with small prime divisors. *Dokl. Akad. Nauk SSSR.* T. 155, p. 1268-1271 (cf. p. 25).

HOOLEY, C., 1964. On the distribution of the roots of polynomial congruences. *Mathematika.* T. 11, nº 01, p. 39-49 (cf. p. 14).

IWANIEC, H., 1978. Almost-primes represented by quadratic polynomials. *Invent. Math.* T. 47, p. 171-188 (cf. p. 24).

IWANIEC, H., 1982. Mean values for Fourier coefficients of cusp forms and sums of Kloosterman sums. In : *Journées arithmétiques 1980.* T. 56, p. 306-321. London Math. Soc. Lecture Note Ser. (Cf. p. 10).

IWANIEC, H., 1997. *Topics in classical automorphic forms.* Providence, RI : American Mathematical Society. Graduate Studies in Mathematics (cf. p. 38).

IWANIEC, H. ; KOWALSKI, E., 2004. *Analytic number theory.* Cambridge Univ Press (cf. p. 36).

KASHAEV, R. M., 1997. The hyperbolic volume of knots from the quantum dilogarithm. *Lett. Math. Phys.* T. 39, nº 3, p. 269-275 (cf. p. 38).

KATZ, N. M., 1980. *Sommes exponentielles. Cours à Orsay, automne 1979. Redige par Gerard Laumon, preface par Luc Illusie.* Astérisque (cf. p. 12).

KATZ, N. M. ; SARNAK, P., 1999. Zeroes of zeta functions and symmetry. *Bull. Am. Math. Soc., New Ser.* T. 36, nº 1, p. 1-26 (cf. p. 21).

KIRAL, E. M. ; YOUNG, M. P., 2019. Kloosterman sums and Fourier coefficients of Eisenstein series. *Ramanujan J.* T. 49, nº 2, p. 391-409 (cf. p. 12, 28).

KLOECKNER, B. R., 2019. Effective perturbation theory for simple isolated eigenvalues of linear operators. *J. Oper. Theory.* T. 81, nº 1, p. 175-194 (cf. p. 31).

KLOOSTERMAN, H. D., 1927. On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$. *Acta Math.* T. 49, p. 407-464 (cf. p. 10).

KOWALSKI, E., 2003. Classical automorphic forms. In : *An introduction to the Langlands program. Lectures presented at the Hebrew University of Jerusalem, Jerusalem, Israel, March 12–16, 2001.* Boston, MA : Birkhäuser, p. 39-71 (cf. p. 14).

KUZNETSOV, N. V., 1981. Petersson's conjecture for cusp forms of weight zero and Linnik's conjecture. Sums of Kloosterman sums. *Math. USSR, Sb.* T. 39, nº 3, p. 299-342 (cf. p. 10, 11).

LA BRETÈCHE, R. ; DRAPPEAU, S., 2020. Level of distribution of quadratic polynomials and an upper bound sieve for friable integers. *J. Eur. Math. Soc. (JEMS).* T. 22, nº 5, p. 1577-1624 (cf. p. 23-25).

LEE, J. ; SUN, H.-S., 2019. *Dynamics of continued fractions and distribution of modular symbols.* Disp. à l'adr. eprint : http://arxiv.org/abs/1902.06277. Prépublication (cf. p. 36).

LINNIK, Ju. V., 1963. *The dispersion method in binary additive problems.* American Mathematical Society (AMS), Providence, RI. Transl. Math. Monogr. (Cf. p. 18).

MAYNARD, J., 2020. *Primes in arithmetic progressions to large moduli I : Fixed residue classes.* À paraître à Mem. Amer. Math. Soc. (cf. p. 11).

MAZUR, B. ; RUBIN, K., 2021. *Arithmetic conjectures suggested by the statistical behavior of modular symbols.* Disp. à l'adr. eprint : https://arxiv.org/abs/1910.12798. À paraître à Exp. Math. (cf. p. 35).

MERIKOSKI, J., 2022. *On the largest prime factor of* n^2+1. À paraître à J. Eur. Math. Soc. (JEMS) (cf. p. 24).

MONTGOMERY, H. L., 1973. *The pair correlation of zeros of the zeta function* [Analytic Number Theory, Proc. Sympos. Pure Math. 24, St. Louis Univ. Missouri 1972, 181-193 (1973).] (cf. p. 21).

MOREE, P., 2014. Integers without large prime factors : from Ramanujan to de Bruijn. *Integers.* T. 14A, Paper No. A5, 13 (cf. p. 25).

NORDENTOFT, A. C., 2021a. A note on additive twists, reciprocity laws and quantum modular forms. *Ramanujan J.* T. 56, nº 1, p. 151-162 (cf. p. 36).

NORDENTOFT, A. C., 2021b. Central values of additive twists of cuspidal L-functions. *J. Reine Angew. Math.* T. 2021, nº 776, p. 255-293 (cf. p. 36).

OHTSUKI, T., 2016. On the asymptotic expansion of the Kashaev invariant of the 5_2 knot. *Quantum Topology.* T. 7, nº 4, p. 669-735 (cf. p. 38).

PETRIDIS, Y. N. ; RISAGER, M. S., 2018. Arithmetic statistics of modular symbols. *Invent. Math.* P. 1-57 (cf. p. 35).

PETROW, I. ; YOUNG, M. P., 2020. The Weyl bound for Dirichlet L-functions of cube-free conductor. *Ann. of Math. (2).* T. 192, nº 2, p. 437-486 (cf. p. 12).

PITT, N., 2013. On an analogue of Titchmarsh's divisor problem for holomorphic cusp forms. *J. Amer. Math. Soc.* T. 26, nº 3, p. 735-776 (cf. p. 10).

PRATT, K. ; ROBLES, N. ; ZAHARESCU, A. et al., 2020. More than five-twelfths of the zeros of ζ are on the critical line. *Res. Math. Sci.* T. 7, nº 1, p. 74. Id/No 2 (cf. p. 10).

RAMARÉ, O., 2013. Prime numbers : emergence and victories of bilinear forms decomposition. *Eur. Math. Soc. Newsl.* Nº 90, p. 18-27 (cf. p. 17).

RUKAVISHNIKOVA, M. G., 2011. The law of large numbers for the sum of the partial quotients of a rational number with fixed denominator. *Math. Notes*. T. 90, nº 3, p. 418-430 (cf. p. 40).

SICA, F., 1998. *The order of vanishing of L-functions at the center of the critical strip*. Thèse de doct. McGill University, Montréal (cf. p. 21).

SUDLER, C., 1964. An estimate for a restricted partition function. *Q. J. Math.* T. 15, p. 1-10 (cf. p. 37).

TANG, H., 2020. A note on Titchmarsh divisor problem under the generalized Riemann hypothesis. *Rocky Mt. J. Math.* T. 50, nº 6, p. 2223-2233 (cf. p. 15).

TITCHMARSH, E. C., 1930. A divisor problem. *Rend. Circ. Mat. Palermo*. T. 54, nº 1, p. 414-429 (cf. p. 15).

VALLÉE, B., 2000. Digits and continuants in Euclidean algorithms. Ergodic versus Tauberian theorems. *J. Théor. Nombres Bordeaux*. T. 12, nº 2, p. 531-570. Colloque International de Théorie des Nombres (Talence, 1999) (cf. p. 29).

VINOGRADOV, I. M., 1937. A new estimate of a certain sum containing primes. *Recueil Mathématique. Nouvelle Série*. T. 2, p. 783-792 (cf. p. 18).

WEIL, A., 1948. On some exponential sums. *Proc. Nat. Acad. Sci. U.S.A.* T. 34, nº 5, p. 204 (cf. p. 10).

WU, X., 2020. *The fourth moment of Dirichlet L-functions at the central value*. Disp. à l'adr. eprint : `https://arxiv.org/pdf/2008.13407.pdf`. Prépublication (cf. p. 27).

XI, P., 2018. Sign changes of Kloosterman sums with almost prime moduli. II. *Int. Math. Res. Notices IMRN*. T. 2018, nº 4, p. 1200-1227 (cf. p. 12).

XI, P., 2020. When Kloosterman sums meet Hecke eigenvalues. *Invent. Math.* T. 220, nº 1, p. 61-127 (cf. p. 14).

YOUNG, M. P., 2011. The fourth moment of Dirichlet L-functions. *Ann. of Math. (2)*. T. 173, nº 1, p. 1-50 (cf. p. 10, 27).

ZACHARIAS, R., 2019. Simultaneous non-vanishing for Dirichlet L-functions. *Ann. Inst. Fourier*. T. 69, nº 4, p. 1459-1524 (cf. p. 12).

ZAGIER, D., 2010. Quantum modular forms. In : *Quanta of maths*. T. 11, p. 659-675. Clay Math. Proc. (Cf. p. 28, 37).