EFFECTIVE ESTIMATION OF SOME OSCILLATORY INTEGRALS RELATED TO INFINITELY DIVISIBLE DISTRIBUTIONS

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ABSTRACT. We present a practical framework to prove, in a simple way, two-terms asymptotic expansions for Fourier integrals

$$\mathcal{I}(t) = \int_{\mathbb{R}} (\mathrm{e}^{it\phi(x)} - 1)\,\mathrm{d}\mu(x)$$

where μ is a probability measure on $\mathbb{R}$ and ϕ is measurable. This applies to many basic cases, in link with Levy's continuity theorem. We present applications to limit laws related to rational continued fractions coefficients.

1. INTRODUCTION

Let μ be a probability measure on $\mathbb{R}$, and $\phi : \mathbb{R} \to \mathbb{R}$ be μ-measurable. The present paper is concerned with asymptotic formulæ for the Fourier integrals associated with ϕ near the origin,

$$\mathcal{I}[\phi](t) := \int (\mathrm{e}^{it\phi(x)} - 1)\,\mathrm{d}\mu(x), \quad (t \to 0). \tag{1.1}$$

Such estimates are connected with the question of whether the push-forward measure $\phi_*(\mu)$ belongs to the bassin of attraction of a stable law, see Chapter 2 of [IL71]. Our interest in this question originates from this point of view, and more specifically from the work [BD] where we study the convergence towards stable laws of the value distribution of invariants related to modular forms. In the setting of [BD], the measure μ is the Gauss-Kuzmin distribution

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{(1+x)\log 2} \qquad (x \in [0,1]),$$

and this measure is invariant under the Gauss map $T(x) = \{1/x\}$, where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. More precisely, in [BD], we are interested in Birkhoff sums

$$\sum_{j=1}^{r} \phi(T^r(x)), \qquad (T^r = T \circ \cdots \circ T), \tag{1.2}$$

where x varies among rationals and $r \geq 0$ is the length of the continued fractions expansion of x. In the set of rationals we consider, these sums are found to typically behave as sums of the shape

$$\sum_{j=1}^{r} \phi(X_r)$$

where $(X_j)_{1 \leq j \leq r}$ are i.i.d. random variables distributed according to the Gauss-Kuzmin measure μ. Then effective estimates for the integral (1.1), in conjunction with [BD, Theorem 3.1]

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and the Berry-Esseen inequality [Fel71, equation (XVI.3.13)] are used to obtain uniform limit theorems for the rational Birkhoff sums (1.2).

We return to the setting where μ is an arbitrary probability measure on $\mathbb{R}$. Integrals (1.1) are related to the methods of asymptotic analysis mentioned *e.g.* in Chapter 9 of the monograph [Olv97]. When expressed as convolution integrals $\int_x h(tx)f(x)\,\mathrm{d}x$, they are refered to as h-transforms in [BH86], and are also the topic of interest of the recent work [Lóp08]. The variety in assumptions and methods seems to prevent us from having a uniform framework for estimating (1.1).

The goal of the present paper is to present and prove several basic estimates through which one can give a streamlined and simple proof of an effective asymptotic expansion of the integral (1.1), including the terms of interest in central limit theorems.

Definition 1.1. *Given $\alpha \in (0,3]$ and two positive functions L, R defined in a neighborhood of 0 in $\mathbb{R}_+^*$, we denote by $\mathcal{G}(\alpha, L, R)$ the set of functions $\phi : \mathbb{R} \to \mathbb{R}$ such that for some numbers $c_1, c_2 \in \mathbb{R}$ and $c_* \in \mathbb{C}$, and all small enough $t > 0$, there holds*

$$\mathcal{I}[\phi](t) = ic_1 t + c_2 t^2 + c_* t^\alpha L(t) + O(t^3 + t^\alpha R(t)). \tag{1.3}$$

Remark. – If $R = O(t^\varepsilon)$ for any $\varepsilon > 0$ and $\alpha < 1$, the term $c_1 t$ in (1.3) is part of the error term, and likewise for $c_2 t^2$ if $\alpha < 2$.

– We will be interested in the largest one or two terms in the expansion (1.3). The case $\alpha = 3$, $L = R \equiv 1$ corresponds to an order 2 Taylor expansion.

– Whenever the expansion (1.3) holds for ϕ, we will denote the coefficients by $c_1(\phi)$, $c_2(\phi)$, $c_*(\phi)$ respectively.

Theorem 1.2. *(1) If $\int |\phi(x)|^\alpha \,\mathrm{d}\mu(x) < \infty$ for some $\alpha \in (0,3]$, then $\phi \in \mathcal{G}(\alpha, 1, 1)$.*

(2) Suppose that $\mathrm{d}\mu = f\,\mathrm{d}\nu$ where ν is the Lebesgue measure and $f \in \mathcal{C}^1([0,1])$. Then for all $a \in \mathbb{R}^*$, $\beta > 3$ and $\lambda \geq 0$, the function

$$\phi : (0,1] \to \mathbb{R}, \qquad \phi(x) = ax^{-\beta}|\log x|^\lambda,$$

belongs to $\mathcal{G}(\frac{1}{\beta}, |\log|^{\lambda/\beta + v}, |\log|^{\lambda/\beta + v - 1 + \varepsilon})$ for any $\varepsilon \in (0,1]$, where $v = 1$ for $\beta \in \{1/2, 1\}$ and $v = 0$ otherwise.

(3) Given two measurable functions ϕ_1, ϕ_2, such that $\phi_j \in \mathcal{G}(\alpha_j, L_j, R_j)$ with $t^{\alpha_2} L_2(t) = O(t^{\alpha_1} L_1(t))$ as $t \to 0$, then $\phi_1 + \phi_2 \in \mathcal{G}(\alpha_1, L_1, R_+)$ for some positive function R_+ explicit in terms of L_1, L_2 and R_1.

The three items here are special cases of Proposition 2.1, Corollary 2.3 and Proposition 2.5 below, respectively. The coefficients c_1, c_2 and c_* and the function R_+ are explicitly described in the precise versions below.

The proofs of all three result are rather short, but together they allow for a simple proof of the expansion (1.1) in several concrete cases:

– In Corollary 3.1, we study a function $\phi : (0,1] \to \mathbb{R}^2$ having an asymptotic behaviour around 0 of the shape $x^{-1/2}|\log x|$. The ensuing estimate we obtain is used in [BD, Theorem 2.1] to deduce a central limit theorem for central values $\{D(1/2, x), x \in \mathbb{Q} \cap (0,1]\}$ of the analytic continuation of the Estermann function

$$D(s,x) = \sum_{n\geq 1} \frac{\tau(n)}{n^s} \mathrm{e}^{2\pi i n x}, \qquad (\mathrm{Re}(s) > 1), \tag{1.4}$$

where τ is the divisor function.

– In Corollaries 3.3 and 3.2, we study the functions of the shape $\phi(x) = \lfloor 1/x \rfloor^\lambda$ where $\lambda \geq 1/2$. These functions occur when studying the values $\{\Sigma_\lambda(x), x \in \mathbb{Q} \cap (0,1]\}$ of the moments of the continued fractions coefficients,

$$\Sigma_\lambda(x) = \sum_{j=1}^r a_j^\lambda, \qquad (x = [0; a_1, \ldots, a_r] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}, a_r > 1),$$

see [BD, Theorems 2.5 and 9.4]. This, in turn, is applied to obtain a law of large numbers for the values of the Kashaev invariants of the 4_1 knot [BD, Corollary 2.6].

- In Corollary 3.4, we study the function ϕ on $(0,1]$ given by $\phi(x) = \lfloor 1/x \rfloor - \lfloor 1/T(x) \rfloor$, where $T : (0,1] \to (0,1]$, $T(x) = \{1/x\}$ is the Gauss map. The estimate we obtain is used in [BD, Theorem 2.7] to obtain an independent proof, using dynamical systems, of a theorem of Vardi [Var93] on the convergence to a Cauchy law of the values of Dedekind sums.

2. Estimation of (1.1) in general

2.1. **Basic estimates.**

2.1.1. *Taylor estimate.* The first and simplest method to obtain an estimate for (1.1) is to insert and integrate a Taylor expansion for the exponential.

Proposition 2.1. *Assume that for some $\alpha \in (0,3]$, we have*

$$K := \int |\phi(x)|^\alpha \,\mathrm{d}\mu(x) < \infty.$$

Then $\phi \in \mathcal{G}(\alpha,1,1)$, and more precisely

$$\mathcal{I}[\phi](t) = ic_1 t + c_2 t^2 + O(Kt^\alpha) \tag{2.1}$$

with $c_1 = \int \phi \,\mathrm{d}\mu$ if $\alpha \geq 1$, and $c_2 = -\frac{1}{2}\int |\phi|^2 \,\mathrm{d}\mu$ if $\alpha \geq 2$. The implied constant is absolute.

Proof. We use the bound $\left|\mathrm{e}^{iu} - \sum_{0\leq k<\alpha} \frac{(iu)^k}{k!}\right| \ll |u|^\alpha$ with $u = t\phi(x)$, and integrate over x. ☐

Although it will not be useful for us here, we note that in the precise bound (2.1), the value of α could be taken as a function of t. For example, if μ is the Lebesgue measure on $(0,1)$ and $\phi(x) = 1/x$, we can take $\alpha = 1 - 1/|\log t|$ and obtain $\mathcal{I}[\phi](t) = O(t|\log t|)$.

2.1.2. *Using properties of the Mellin transform.* When the moment $\int |\phi|^\alpha \,\mathrm{d}\mu$ diverges at some particular α, we can often extract a useful expansion from the Cauchy formula and the polar behaviour of the Mellin transform. For $x \in \mathbb{R}$, $s \in \mathbb{C}$ and $\eta \in [0,1]$, let

$$\phi_{s,\eta}(x) := \mathbf{1}_{\phi(x)\neq 0}|\phi(x)|^s \exp(-s\tfrac{\pi i}{2}(1-\eta)\operatorname{sgn}\phi(x)), \qquad \phi_s(x) := \phi_{s,0}(x).$$

Note that for $k \in \mathbb{N}_{>0}$, $\phi_k(x) = (-i\phi(x))^k$. Define further

$$G_\eta(s) := \int \phi_{s,\eta}(x)\,\mathrm{d}\mu(x).$$

Proposition 2.2. *Let $\alpha \in (0,3)$, $\rho \in (0,1)$, $\delta, \eta_0 > 0$ and $\xi \in \mathbb{R}$. Assume that for some $c > 0$, we have*

$$\int_{\phi(x)\neq 0} (|\phi(x)|^c + |\phi(x)|^{-c})\,\mathrm{d}\mu(x) < \infty \tag{2.2}$$

and that the functions $G_\eta(s)$ for $\eta \in [0,\eta_0]$, initially defined for $\mathrm{Re}(s) \in (-c,c)$, can be analytically continued to the set

$$\{s \in \mathbb{C}, 0 < \mathrm{Re}(s) \leq \alpha+\delta, s \notin [\alpha,\alpha+\delta]\}.$$

Assume further that

$$\sup_{0\leq\eta\leq\eta_0} \int_{\substack{\tau\in\mathbb{R}\\ s=\alpha+\delta+i\tau}} |\Gamma(-s)G_\eta(s)|\,\mathrm{d}\tau < \infty,$$

and that there is an open neighborhood $\mathcal{V}$ of $[\alpha,\alpha+\delta]$ for which

$$(\alpha-s)^\xi G_0(s) = \varrho + O(|s-\alpha|^\rho), \qquad s \in \mathcal{V}\smallsetminus[\alpha,\alpha+\delta],\ \mathrm{Re}(s) \leq \alpha+\delta. \tag{2.3}$$

Then, $\phi \in \mathcal{G}(\alpha, |\log|^{\xi-1+\upsilon_\alpha}, |\log|^{\xi-1+\upsilon_\alpha-\rho})$, where $\upsilon_\alpha = 1$ if $\alpha = 1, 2$ and $\upsilon_\alpha = 0$ otherwise, and with coefficients given by

$$c_1 = iG_0(1) \text{ if } \alpha > 1, \qquad c_2 = \tfrac{1}{2}G_0(2) \text{ if } \alpha > 2, \qquad c_* = \begin{cases} -\varrho/\Gamma(\xi+1), & \alpha = 1, \\ \frac{1}{2}\varrho/\Gamma(\xi+1), & \alpha = 2, \\ \varrho\frac{\Gamma(-\alpha)}{\Gamma(\xi)}, & \alpha \notin \{1,2\}. \end{cases} \tag{2.4}$$

Proof. We write

$$\mathcal{I}[\phi](t) + 1 = \int \mathrm{e}^{it\phi(x)}\,\mathrm{d}\mu(x) = J_+ + J_- + J_0,$$

where $J_\pm$ corresponds to the part of the integral restricted to $\pm\phi > 0$.

For all $\varepsilon \in (0, \frac{\pi}{2}\eta_0)$, define

$$J_+(\varepsilon) := \int_{\phi(x)>0} \mathrm{e}^{(-\varepsilon+i)t\phi(x)}\,\mathrm{d}\mu(x), \qquad J_-(\varepsilon) := \int_{\phi(x)<0} \mathrm{e}^{(\varepsilon+i)t\phi(x)}\,\mathrm{d}\mu(x).$$

By dominated convergence, we have $J_+ := \lim_{\varepsilon\to0^+} J_+(\varepsilon)$, and similarly for J_-. We use the Mellin transform formula for the exponential

$$\mathrm{e}^{-y} = \frac{1}{2\pi i}\int_{-c/2-i\infty}^{-c/2+i\infty} \Gamma(-s)|y|^s\mathrm{e}^{s\arg(y)}\,\mathrm{d}s$$

valid for $\mathrm{Re}(y) > 0$, see [GR07, eq. 17.43.1] (the extension to non-real y is straightforward by the Stirling formula [GR07, eq. 8.327.1]). Inserting this in $J_\pm(\varepsilon)$, we obtain

$$J_+(\varepsilon) + J_-(\varepsilon) = \frac{1}{2\pi i}\int_{-c/2-i\infty}^{-c/2+i\infty} \Gamma(-s)G_\eta(s)|1+i\varepsilon|^s t^s\,\mathrm{d}s,$$

where $\eta = \frac{2}{\pi}\arctan\varepsilon \le \frac{2\varepsilon}{\pi} \le \eta_0$. We move the contour forward to $\mathrm{Re}(s) = \alpha + \delta$. The simple pole at $s = 0$ contributes $\int_{\phi(x)\neq0}\mathrm{d}\mu(x)$, and therefore by adding the contribution from J_0 we get

$$\begin{aligned} J_0 + J_+(\varepsilon) + J_-(\varepsilon) = 1 + R &+ \frac{1}{2\pi i}\int_{H(\alpha,\alpha+\delta)} \Gamma(-s)G_\eta(s)t^s|1+i\varepsilon|^s\,\mathrm{d}s \\ &+ \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\alpha+\delta} \Gamma(-s)G_\eta(s)t^s|1+i\varepsilon|^s\,\mathrm{d}s, \end{aligned}$$

where R consists of the contribution of the residues at 1 (if $\alpha > 1$) and 2 (if $\alpha > 2$). Here $H(\alpha, \alpha+\delta)$ is a Hankel contour, going from $\alpha+\delta-i0$ to $\alpha+\delta+i0$ passing around α from the left. The last integral is bounded by the triangle inequality, using our first hypothesis on G_η, which gives

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\alpha+\delta} \Gamma(-s)G_\eta(s)t^s|1+i\varepsilon|^s\,\mathrm{d}s \ll t^{\alpha+\delta},$$

uniformly in ε. Passing to the limit $\varepsilon \to 0$, there remains to prove

$$\frac{1}{2\pi i}\int_{H(\alpha,\alpha+\delta)} \Gamma(-s)G_0(s)t^s\,\mathrm{d}s = c_* t^\alpha|\log t|^{\xi-1+\upsilon_\alpha} + O(t^\alpha|\log t|^{\xi-1+\upsilon_\alpha-\rho}).$$

This is done by using our second hypothesis along with a standard Hankel contour integration argument; we refer to *e.g.* Corollary II.0.18 of [Ten15] for the details. □

An important special case is the following.

Corollary 2.3. *Let μ be defined on $[0,1]$ by $\mathrm{d}\mu(x) = f(x)\,\mathrm{d}x$ where $f \in \mathcal{C}^1([0,1])$. Let $a \in \mathbb{R}\smallsetminus\{0\}$. For all $\beta > \frac{1}{3}$, $\lambda \ge 0$, and ϕ given by*

$$\phi(x) = ax^{-\beta}|\log x|^\lambda$$

one has $\phi \in \mathcal{G}(1/\beta, |\log|^{\lambda/\beta+v_{1/\beta}}, |\log|^{\lambda/\beta+v_{1/\beta}-1+\varepsilon})$ *for any* $\varepsilon \in (0,1)$ *and with*

$$c_* = f(0)\frac{|a|^{1/\beta}e^{\frac{-\pi i\,\mathrm{sgn}\,a}{2\beta}}}{\beta^{\lambda/\beta+1}} \times \begin{cases} -(\lambda+1)^{-1}, & \beta = 1, \\ (4\lambda+2)^{-1}, & \beta = 1/2, \\ \Gamma(-1/\beta), & \beta \notin \{1, 1/2\}. \end{cases}$$

and $c_1 = \int \phi \,\mathrm{d}\mu$ *if* $\beta < 1$ *and* $c_2 = -\frac{1}{2}\int |\phi|^2 \,\mathrm{d}\mu$ *if* $\beta < \frac{1}{2}$.

Proof. First, we write $\mathrm{d}\mu(x) = f(0)\chi(x)\,\mathrm{d}x + xg(x)\,\mathrm{d}x$, where χ is the characteristic function of the interval $[0,1]$ and $g \in \mathcal{C}([0,1])$. For the contribution of $\chi\,\mathrm{d}x$ we apply Proposition 2.2 with any fixed $c < 1/\beta$, $\alpha = 1/\beta$, $\xi = \lambda/\beta + 1$, any fixed $\rho \in (0,1)$ and $\delta > 0$. By [GR07, 4.272.6], for $\mathrm{Re}(s) < 1/\beta$ and $\eta \in [0,1]$ we have

$$G_\eta(s) = e^{-s\frac{\pi i}{2}(1-\eta)\,\mathrm{sgn}(a)}|a|^s \int_0^1 x^{-\beta s}|\log x|^{\lambda s}\,\mathrm{d}x = e^{-s\frac{\pi i}{2}(1-\eta)\,\mathrm{sgn}(a)}|a|^s \frac{\Gamma(\lambda s+1)}{(1-\beta s)^{\lambda s+1}}.$$

Notice also that by Stirling's formula $G_\eta(s) \ll \mathrm{e}^{\pi(\frac{1-\eta}{2})|\tau|}|\tau|^{-1/2}$ as $|\tau| = |\operatorname{Im} s| \to \infty$, so that in any case $\Gamma(-s)G_\eta(s) \ll |\tau|^{-1-\mathrm{Re}(s)}$. Therefore the hypotheses of Proposition 2.2 are easily verified with

$$\varrho = |a|^{1/\beta}e^{\frac{-\pi i\,\mathrm{sgn}\,a}{2\beta}}\frac{\Gamma(\lambda/\beta+1)}{\beta^{\lambda/\beta+1}}.$$

Thus,

$$\int_0^1 (\mathrm{e}^{it\phi(x)}-1)\,\mathrm{d}x = itc_1' + c_2't^2 + c_*t^{1/\beta}|\log t|^{\lambda/\beta+v_{1/\beta}} + O(t^{1/\beta}|\log t|^{\lambda/\beta+v_{1/\beta}-\rho})$$

with coefficients as given in (2.4) with $G_0(1) = -i\int \phi\chi\,dx$ and $G_0(2) = -\int \phi^2\chi\,dx$. Finally, as in Proposition 2.1 we deduce

$$\int (\mathrm{e}^{it\phi(x)}-1)xg(x)\,\mathrm{d}x = ic_1''t + c_2''t^2 + O(Kt^{\alpha'})$$

for any $0 < \alpha' < \min(3, \frac{2}{\beta})$ and with $c_1'' = \int \phi(x)xg(x)\,\mathrm{d}x$ if $\alpha' > 1$ and $c_2'' = -\frac{1}{2}\int \phi(x)^2xg(x)\,\mathrm{d}x$ if $\alpha' > 2$. The result then follows. □

2.2. **Addition.**

Lemma 2.4. *For* $j \in \{1,2\}$, *let* $\delta_j(x) = \mathrm{e}^{it\phi_j(x)} - 1$. *Then*

$$\begin{aligned} \mathcal{I}[\phi_1+\phi_2](t) &= \mathcal{I}[\phi_1](t) + \mathcal{I}[\phi_2](t) + \int \delta_1(x)\delta_2(x)\,\mathrm{d}\mu(x) \\ &= \mathcal{I}[\phi_1](t) + \mathcal{I}[\phi_2](t) + O\Big(\prod_{j\in\{1,2\}} |\mathrm{Re}\,\mathcal{I}[\phi_j](t)|^{1/2}\Big) \end{aligned} \tag{2.5}$$

Proof. The first equation is simply the relation $\mathrm{e}^{it(\phi_1(x)+\phi_2(x))} - 1 = \delta_1(x) + \delta_2(x) + \delta_1(x)\delta_2(x)$ integrated over x. The last term is bounded using the Cauchy-Schwarz inequality

$$\Big(\int |\delta_1(x)\delta_2(x)|\,\mathrm{d}\mu(x)\Big)^2 \le \prod_{j\in\{1,2\}} \int |\delta_j(x)|^2\,\mathrm{d}\mu(x)$$

and expanding the square on the right-hand side. □

Proposition 2.5. *For* $j \in \{1,2\}$, *let* $\alpha_j \in (0,2]$, *let* L_j, R_j *be positive functions defined on a neighborhood of* 0 *in* $\mathbb{R}_+^*$, *and* $\phi_j \in \mathcal{G}(\alpha_j, L_j, R_j)$. *If* $\alpha_1 \le \alpha_2$, *and under the following assumptions:*

– $R_j(t), L_j(t) = t^{o(1)}$ *as* $t \to 0$,
– $R_j(t) = O(L_j(t))$,
– $t^2 = O(t^{\alpha_1}L_1(t))$,

we have

$$\phi_1 + \phi_2 \in \mathcal{G}(\alpha_1, L_1, R_+), \qquad R_+ = \begin{cases} R_1 & \text{if } \alpha_1 < \alpha_2, \\ R_1 + L_2 + \sqrt{L_1 L_2} & \text{if } \alpha_1 = \alpha_2 < 2, \\ R_1 + L_2 + \sqrt{L_1}(\sqrt{L_2} + 1) & \text{if } \alpha_1 = \alpha_2 = 2. \end{cases}$$

Moreover,

$$c_1(\phi_1 + \phi_2) = c_1(\phi_1) + c_1(\phi_2),$$
$$c_*(\phi_1 + \phi_2) = c_*(\phi_1).$$

Proof. We use Lemma 2.4; when computing the real part in (2.5), the term $ic_1 t$ vanishes. □

Remark. Note that using this result might induce a slight quantitative loss in the two cases when $\alpha_1 = \alpha_2$. What is gained at this price is that we are only required to study each ϕ_j separately, which simplifies the analysis.

We also remark that this estimate is useful only when the term $c_2 t^2$ is not relevant in (1.3). In the complementary case, Proposition 2.1 can be used, although the ensuing error term will typically be worse than optimal by a factor of $|\log t|$.

It is straightforward to generalize Proposition 2.5, affecting to each ϕ_j a different value of the frequency: under the same hypotheses and notations, and additionally that L_j, R_j tend monotonically to $+\infty$ at 0,

$$\int \mathrm{e}^{it_1\phi_1(x)+it_2\phi_2(x)}\,\mathrm{d}\mu(x) = 1 + ic_1(\phi_1)t_1 + ic_1(\phi_2)t_2 + c_* t_1^{\alpha_1} L_1(t_1) + O(t_+^2 + t_+^{\alpha_1} R_+(t_+)),$$

where c_1, c_* are as in the conclusion of Proposition 2.5, and $t_+ = \max\{t_1, t_2\}$.

3. Applications

We now describe the applications we will be interested in. The measure is the Gauss-Kuzmin distribution

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{(1+x)\log 2} \qquad (x \in [0,1]).$$

The measure μ is invariant under the Gauss map $T(x) = \{1/x\}$ on $(0,1)$, in particular,

$$\mathcal{I}[\phi \circ T](t) = \mathcal{I}[\phi](t). \tag{3.1}$$

3.1. **Central values of the Estermann function.** The first application we discuss is the "period function" $\phi : \mathbb{R} \to \mathbb{C}$ associated with the Estermann function (1.4), namely

$$\phi(x) = D(\tfrac12, 1/x) - D(\tfrac12, x),$$

initially defined in $\mathbb{Q} \cap (0,1]$. By [Bet16], this function can be extended to a continuous function on $(0,1]$, more precisely given by an expression of the shape (3.2) below. Interpreting ϕ to be $\mathbb{R}^2$-valued, the analogue of the integral (1.1) is estimated using the following.

Corollary 3.1. *Let $\varepsilon > 0$, $\mathcal{E} : [0,1] \to \mathbb{C}$ be a bounded, continuous function, and*

$$\boldsymbol{\phi}_j(x) := \begin{pmatrix} \frac12 x^{-1/2}\big(\log(1/x) + \gamma_0 - \log(8\pi) - \frac{\pi}{2}\big) + \zeta(\frac12)^2 + \operatorname{Re}\mathcal{E}((-1)^j x) \\ \frac{(-1)^{j-1}}{2} x^{-1/2}\big(\log(1/x) + \gamma_0 - \log(8\pi) + \frac{\pi}{2}\big) + \operatorname{Im}\mathcal{E}((-1)^j x) \end{pmatrix}. \tag{3.2}$$

Let also $\boldsymbol{u}_j := \binom{1}{(-1)^{j-1}}$. Then for some vector $\boldsymbol{\mu} \in \mathbb{R}^2$, and all $\boldsymbol{t} \in \mathbb{R}^2$, we have

$$\int_0^1 \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_1(x) + \boldsymbol{\phi}_2(T(x))\rangle}\,\mathrm{d}\mu(x)$$
$$= 1 + i\langle \boldsymbol{t}, \boldsymbol{\mu}\rangle - \frac{1}{3\log 2} \sum_{j\in\{1,2\}} \langle \boldsymbol{t}, \boldsymbol{u}_j\rangle^2 |\log|\langle \boldsymbol{t}, \boldsymbol{u}_j\rangle||^3 + O_\varepsilon(\|\boldsymbol{t}\|^2 |\log\|\boldsymbol{t}\||^{2+\varepsilon}).$$

Proof. Let $\varepsilon \in (0,1)$. Using Corollary 2.3 with $\beta = 1/2$ and $\lambda \in \{0,1\}$, and Proposition 2.1, we obtain

$$(x \mapsto \pm\tfrac{1}{2}x^{-1/2}|\log x|) \in \mathcal{G}(2, |\log|^3, |\log|^{2+\varepsilon}),$$
$$(x \mapsto (\gamma_0 - \log(8\pi) + \tfrac{\pi}{2})x^{-1/2}) \in \mathcal{G}(2, |\log|, |\log|^{\varepsilon}),$$
$$(x \mapsto \operatorname{Im}\mathcal{E}(\pm x)) \in \mathcal{G}(3,1,1),$$

as well as $c_*(x \mapsto \pm\frac{1}{2}x^{-1/2}|\log x|) = -\frac{1}{3\log 2}$. From Proposition 2.5 and the ensuing remark, and using the property (3.1), we obtain for $j \in \{1,2\}$

$$\int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_j(x)\rangle} - 1)\,\mathrm{d}\mu(x) = i\langle \boldsymbol{t}, \boldsymbol{\mu}_j\rangle + c_*\langle \boldsymbol{t}, \boldsymbol{u}_j\rangle^2 |\log|\langle \boldsymbol{t}, \boldsymbol{u}_j\rangle||^3 + O_\varepsilon(\|\boldsymbol{t}\|^2 |\log\|\boldsymbol{t}\||^{2+\varepsilon}),$$

where $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathbb{R}^2$. On the other hand, we have

$$\Delta(\boldsymbol{t}) := \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_1(x)\rangle} - 1)(\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_2(T(x))\rangle} - 1)\,\mathrm{d}\mu(x) = \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_2(x)\rangle} - 1)F_x(\boldsymbol{t})\,\mathrm{d}x,$$

where

$$F_x(\boldsymbol{t}) = \frac{1}{\log 2}\sum_{n\geq 1} \frac{\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_1(1/(n+x))\rangle} - 1}{(n+x)(n+x+1)}.$$

By a Taylor expansion at order 1, we have $|F_x(\boldsymbol{t})| \ll \|\boldsymbol{t}\|$ uniformly in x, and therefore

$$|\Delta(\boldsymbol{t})| \ll \|\boldsymbol{t}\|^2 \int_0^1 \|\boldsymbol{\phi}_2(x)\|\,\mathrm{d}x \ll \|\boldsymbol{t}\|^2.$$

By (2.5), we deduce

$$\int_0^1 \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_1(x) + \boldsymbol{\phi}_2(T(x))\rangle}\,\mathrm{d}\mu(x) = 1 + \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_1(x)\rangle} + \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_2(T(x))\rangle} - 2)\,\mathrm{d}\mu(x) + O(\|\boldsymbol{t}\|^2),$$

whence the claimed estimate. ☐

3.2. **Moments of continued fractions coefficients.** The next application we consider pertains to the moments functions Σ_λ of continued fractions coefficients, where $\lambda \geq 0$ is the order of the moment. The function of interest to us here is

$$\phi_\lambda(x) = \lfloor 1/x \rfloor^\lambda.$$

The case $\lambda < 1/2$ can be easily dealt with using Proposition 2.1, so we do not focus on it here.

A first approach is to use Proposition 2.5 to approximate $\lfloor 1/x \rfloor$ by $1/x$, and then use Corollary 2.3. This leads to the following.

Corollary 3.2. *Let $\lambda \geq 1/2$. The function ϕ_λ given by $\phi_\lambda(x) = \lfloor 1/x \rfloor^\lambda$ satisfies the following.*

– *If $\lambda = 1/2$, then with $c_* = -1/(\log 2)$, we have*

$$\mathcal{I}[\phi_{1/2}](t) = ic_1 t + c_* t^2 |\log t| + O_\varepsilon(t^2 |\log t|^\varepsilon). \tag{3.3}$$

– *If $\lambda > 1/2$ and $\lambda \neq 1$, then with $c_* = -\exp(-\pi i/(2\lambda))\Gamma(1 - 1/\lambda)/\log 2$, we have*

$$\mathcal{I}[\phi_\lambda](t) = (\mathbf{1}_{\lambda<1}) ic_1 t + c_* t^{1/\lambda} + O_\varepsilon(t^{1/\lambda}|\log t|^{-1+\varepsilon})$$

When $1/2 \leq \lambda < 1$, we have $c_1 = \int_0^1 \phi_\lambda(x)\,\mathrm{d}\mu(x)$.

Proof. We write $\phi_\lambda(x) = p_\lambda(x) + r_\lambda(x)$, where $p_\lambda(x) = x^{-\lambda}$ and $r_\lambda(x) \ll_\lambda \lfloor 1/x \rfloor^{\lambda - 1}$. By Proposition 2.1, we have $r_\lambda \in \mathcal{G}(\min(3, \frac{1}{\lambda - 1/3}), 1, 1)$.

We consider first the case $\lambda > 1/2$, $\lambda \neq 1$. By Corollary 2.3, we have $p_\lambda \in \mathcal{G}(\frac{1}{\lambda}, 1, |\log|^{-1+\varepsilon})$. We deduce, by Proposition 2.5, that $\phi_\lambda \in \mathcal{G}(\frac{1}{\lambda}, 1, |\log|^{-1+\varepsilon})$, and this yields the second and third cases.

If $\lambda = 1/2$, then Corollary 2.3 implies $p_{1/2} \in \mathcal{G}(2, |\log|, |\log|^\varepsilon)$, and by Proposition 2.1, for some $c \in \mathbb{R}$, we have

$$\mathcal{I}[r_{1/2}](t) = ict + O(t^2)$$

On the other hand, since $\left|(\mathrm{e}^{itp_{1/2}(x)}-1)(\mathrm{e}^{itr_{1/2}(x)}-1)\right| \ll t^2\left|p_{1/2}(x)r_{1/2}(x)\right| \ll t^2$, we get

$$\int_0^1 (\mathrm{e}^{itp_{1/2}(x)}-1)(\mathrm{e}^{itr_{1/2}(x)}-1)\,\mathrm{d}\mu(x) = O(t^2).$$

By (2.5), we conclude (3.3) as claimed. □

The case $\lambda = 1$ could be analyzed by the same method, but we chose to study it separately to obtain a more precise error term by another approach, using Proposition 2.2 directly. The associated Mellin transform $G_0(s)$ is related to the Riemann ζ-function.

Corollary 3.3. *The function ϕ given by $\phi(x) = \lfloor 1/x \rfloor$ satisfies*

$$\mathcal{I}[\phi](t) = -\tfrac{it}{\log 2}(\log t + \gamma_0 - \tfrac{\pi i}{2}) + O_\varepsilon(t^{2-\varepsilon}).$$

Proof. The integral (2.2) converges for all $c < 1$. A quick computation shows that an analytic continuation of $G_\eta(s)$ is given by

$$G_\eta(s) = \frac{\exp(-s\frac{\pi i}{2}(1-\eta))}{\log 2}\{\zeta(2-s) + H(s)\},$$

where $H(s) = \sum_{n\geq 1} n^s(\log(1+\frac{1}{n(n+2)}) - \frac{1}{n^2})$ is analytic and uniformly bounded in $\mathrm{Re}(s) \leq 2-\varepsilon$. We have

$$\int_{\mathrm{Re}(s)=2-\varepsilon} |\Gamma(-s)G_\eta(s)||\mathrm{d}s| \ll_\varepsilon 1 + \int_0^\infty |\zeta(\varepsilon + i\tau)|\frac{\mathrm{d}\tau}{1+\tau^2} \ll_\varepsilon 1$$

by the Stirling formula. The polar behaviour (2.3) is given by

$$G_0(s) = \frac{\exp(-s\frac{\pi i}{2})}{\log 2}\{\zeta(2-s)+H(s)\} = \frac{\exp(-s\frac{\pi i}{2})}{\log 2}\Big\{\frac{1}{1-s} + A + O(s-1)\Big\}$$

for s in a neighborhood of 1, where

$$\begin{aligned} A &= \sum_{n\geq 1}\Big(n\log\Big(1+\frac{1}{n(n+2)}\Big) - \log\Big(1+\frac{1}{n}\Big)\Big) \\ &= -\lim_{N\to\infty}\sum_{n=1}^N \Big(n\log\Big(1+\frac{1}{n+1}\Big) - (n-1)\log\Big(1+\frac{1}{n}\Big)\Big) \\ &= -1. \end{aligned}$$

Applying Proposition 2.2 with $\delta = 1/2$ and $\alpha = 1$ yields the claimed result up to $O(t)$. Our more precise statement follows from noting that there is no branch cut along $s \geq 1$ in this case, so that the residue theorem may be used. We obtain

$$\operatorname*{Res}_{s=1} \Gamma(-s)G_0(s)t^s = \tfrac{it}{\log 2}(\gamma_0 - \tfrac{\pi i}{2} + \log t),$$

whence the claimed estimate. One could go further, isolating a pole of order 2 at $s = 2$, and this would give an error term $O(t^2|\log t|)$. □

3.3. **Dedekind sums.** The final example we discuss is related to Dedekind sums, for the definition of which we refer to [BD, Section 2.4]. The "period function" ϕ relevant to us here is

$$\phi(x) = \lfloor 1/x \rfloor - \lfloor 1/T(x) \rfloor.$$

Compared with the case of $x \mapsto \lfloor 1/x \rfloor$ studied in Corollary 3.3, the relevant exponent α is again 1, but the leading term turns out to be t (the terms $t\log t$ vanish).

Corollary 3.4. *The map ϕ on $(0,1)$ given by $\phi(x) = \lfloor 1/x \rfloor - \lfloor 1/T(x) \rfloor$ satisfies*

$$\mathcal{I}[\phi](t) = -\frac{\pi}{\log 2}t + O(t^2|\log t|^2).$$

Proof. We consider

$$\begin{aligned}\Delta(t) &:= \int_0^1 (\mathrm{e}^{-it\lfloor 1/T(x)\rfloor} - 1)(\mathrm{e}^{it\lfloor 1/x\rfloor} - 1)\,\mathrm{d}\mu(x) \\ &= \int_0^1 (\mathrm{e}^{-it\lfloor 1/x\rfloor} - 1)F_x(t)\,\mathrm{d}x,\end{aligned}$$

with $F_x(t) = \frac{1}{\log 2}\sum_{n\geq 1}\frac{\mathrm{e}^{itn}-1}{(n+x)(n+1+x)}$. Since $|\mathrm{e}^{iu} - 1| \ll |u|^{1-1/|\log t|}$ for all $u \in \mathbb{R}$, we find

$$F_x(t) \ll t\sum_{n\geq 1}\frac{1}{n^{1+1/|\log t|}} \ll t|\log t|.$$

Similarly,

$$\int_0^1 \left|\mathrm{e}^{-it\lfloor 1/x\rfloor} - 1\right|\mathrm{d}x \ll t\int_0^1 x^{-1+1/|\log t|}\,\mathrm{d}x \ll t|\log t|.$$

We thus obtain $\Delta(t) = O((t\log t)^2)$. Using Corollary 3.3 with the improved error term $O(t^2|\log t|)$, (3.1) and (2.5), we deduce

$$\int_0^1 \mathrm{e}^{it(\lfloor 1/x\rfloor - \lfloor 1/T(x)\rfloor)}\,\mathrm{d}\mu(x) = 1 + 2\,\mathrm{Re}\,I(t) + O((t\log t)^2),$$

where $I(t) = \int_0^1(\mathrm{e}^{it\lfloor 1/x\rfloor} - 1)\,\mathrm{d}\mu(x)$. Corollary 3.3 allows us to conclude. □

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