EXPONENTIAL SUMS OVER INTEGERS WITHOUT LARGE PRIME DIVISORS

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Abstract. We obtain a new bound on exponential sums over integers without large prime divisors, improving that of Fouvry and Tenenbaum (1991). For a fixed integer $\nu \neq 0$, we also obtain new bounds on exponential sums with ν-th powers of such integers. The improvement is based on exploiting more precisely the factorisation of integers without large prime divisors, along with existing Type I and Type II bounds. For $\nu = 1$ we use the classical bounds of Vinogradov (1937), while for $\nu \neq 1$ we use bounds of Vaughan (1975) as well as of Fouvry, Kowalski and Michel (2014).

Contents

1. Introduction

Let $P(n)$ denote the largest prime divisor of an integer $n \geqslant 1$, with the convention that $P(1) = 1$.

We recall that an integer n is called y-*smooth* or y-*friable* if $P(n) \leqslant y$, see [16, 20] for a background.

For $x \geqslant y \geq 2$, we consider the set

$$\mathcal{S}(x, y) = \{n \in [1, x] \cap \mathbb{Z} : \ P(n) \leqslant y\}$$

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the set of y-smooth positive integers $n \leqslant x$ and as usual we use $\Psi(x,y) = \#\mathcal{S}(x,y)$ to denote its cardinality.

For real numbers $x \geqslant y \geqslant 2$ and ϑ we define the exponential sum

$$T_\vartheta(x,y) = \sum_{n \in \mathcal{S}(x,y)} \mathbf{e}_q(\vartheta n),$$

where $\mathbf{e}(z) = \exp(2\pi i z)$. Sums of these kind, and their generalisations with non-linear functions of n, have a long history of studying, in particular in relation to Waring's problem and the circle method, see for example [2–4, 6, 10, 13, 17, 26] and the references therein.

The bound of Fouvry and Tenenbaum [13, Theorem 13] (after replacing some logarithmic factors with $x^{o(1)}$) asserts that for $y \leqslant x^{1/2}$, uniformly over integers a and q with $\gcd(a,q) = 1$, we have

$$|T_\vartheta(x,y)| \leqslant x^{1+o(1)} \left(x^{-1/4} y^{1/2} + q^{-1/2} + (x/qy)^{-1/2}\right) \mathscr{L}, \tag{1.1}$$

where

$$\mathscr{L} = 1 + x\left|\vartheta - \frac{a}{q}\right|. \tag{1.2}$$

In turn, the bound (1.1) follows from the bound

$$|S_{a,q}(x,y)| \leqslant x^{1+o(1)} \left(x^{-1/4} y^{1/2} + q^{-1/2} + (x/qy)^{-1/2}\right) \tag{1.3}$$

on rational sums

$$S_{a,q}(x,y) = T_{a/q}(x,y) = \sum_{n \in \mathcal{S}(x,y)} \mathbf{e}_q(an),$$

where $\mathbf{e}_q(z) = \exp(2\pi i z/q)$.

It is easy to see that the bound (1.3) is trivial unless $x \geqslant q^{1+\varepsilon} y$ and $y \leqslant x^{1/2-\varepsilon}$ for some fixed $\varepsilon > 0$. Here, we exploit more precisely the bilinear structure of the exponential sums, and obtain a bound which is nontrivial starting from $x \geqslant q^{1+\varepsilon}$ which is clearly an optimal range (apart from the presence of $\varepsilon > 0$).

In order to simplify the exposition, we concentrate on the regime when y grows as some power x, say $y = x^{\eta+o(1)}$, for some fixed $\eta > 0$, and in particular we have $\Psi(x,y) \geqslant c(\eta)x$, for some constant $c(\eta) > 0$, depending only on η, see [16, 20]. Hence, in this range, the trivial bound $T_\vartheta(x,y)$, which we try to improve, is essentially $|T_\vartheta(x,y)| \leqslant x$. A more careful examination of our argument, with full book-keeping of all logarithmic factors and invoking better bounds on the divisor function "on average", is most likely able to lead to new bounds also in the range when $y = x^{o(1)}$.

Theorem 1.1. *Let $\varepsilon > 0$. For all $x \geqslant y \geqslant 2$, and all integers a with* $\gcd(a,q) = 1$, *we have*

$$|S_{a,q}(x,y)| \leqslant x^{1+o(1)} \left(\min\{x^{-1/5}, (x/y)^{-1/4}\} + q^{-1/2} + (x/q)^{-1/2}\right).$$

The saving $x^{-1/5}$ corresponds to the classical Vinogradov error term $x^{4/5}$ for exponential sums over primes [33].

Using partial summation, as in the proof of [13, Theorem 13], we now estimate the sums $T_\vartheta(x,y)$.

Corollary 1.2. *Uniformly for $x \geqslant y \geqslant 2$ and all real ϑ, we have*

$$|T_\vartheta(x,y)| \leqslant x^{1+o(1)} \left(\min\{x^{-1/5}, (x/y)^{-1/4}\} + q^{-1/2} + (x/q)^{-1/2}\right) \mathscr{L},$$

where $\mathscr{L}$ is given by (1.2).

A variety of other bounds can be found in [2–4, 6, 10, 17, 26], which improve (1.1) and (1.3) (especially for small y), however they do not seem to affect our improvement.

We also recall that for the classical Waring problem and also similar Waring type problems exponential sums with powers n^ν have also been considered, see [7, 8, 11, 31, 35, 36] and references therein. For rational exponential sums with other non-linear functions over the integers $n \in \mathcal{S}(x,y)$ see [15, 28]. Motivated by these results we consider the rational exponential sums

$$S_{\nu,a,q}(x,y) = \sum_{n \in \mathcal{S}(x,y)} \mathbf{e}_q(an^\nu)$$

with ν-th powers of smooth numbers, where $\nu \in \mathbb{Z} \setminus \{0\}$. Combining our approach to proving Theorem 1.1 with some bounds from [25] we obtain new estimates on these sums too. We however have to assume that q is prime.

First we observe that the bound (1.3) can be extended to the sums $S_{\nu,a,q}(x,y)$ (at least for a prime q, and to be nontrivial this generalisation still requires $x \geqslant q^{1+\varepsilon} y$ with some fixed $\varepsilon > 0$). We obtain a bound which gives a power saving for smaller values of x.

Theorem 1.3. *Let $\nu \neq 0$ be a fixed positive integer, and let $\varepsilon > 0$. There exists $\delta > 0$, which depends only on ν and ε and such that the following holds. Assume that a prime $q \geqslant 1$ and real $x \geqslant y \geqslant 2$ satisfy $q \leqslant x^2$. Then, uniformly over a with* $\gcd(a,q) = 1$, *we have the estimates:*

$$S_{\nu,a,q}(x,y) \leqslant x^{1+o(1)} \min\{E_1, E_2, E_3, E_4\},$$

where

$$E_1 = (x/y)^{-1/4} + q^{-1/2} + (x/q)^{-1/2}, \tag{1.4a}$$

$$E_2 = y^{-1/2} + x^{-1/4}q^{1/8} + q^{-1/2} + (x/q)^{-1/2}, \tag{1.4b}$$

$$E_3 = \min\left\{(x/q)^{-1/4}, (x/y)^{-1/4}q^{1/8}\right\} + q^{-1/4} + (x/y)^{-1/4}, \tag{1.4c}$$

$$E_4 = \left(q^{-1/4} + q^{3/4+\varepsilon}x^{-1}\right)^{\delta}. \tag{1.4d}$$

The estimates (1.4a) and (1.4b) are non-trivial only inside to $q < x$, but are numerically better in most of that range. The bounds (1.4c) and (1.4d) hold true upon replacing the function $n \mapsto \mathbf{e}_q(an^\nu)$ by any non-exceptional trace function, in the terminology of [14], unlike the estimates (1.4a) and (1.4b) which use the morphism property of monomials. The regions where these bounds are non-trivial are drawn in Figure 1.1.

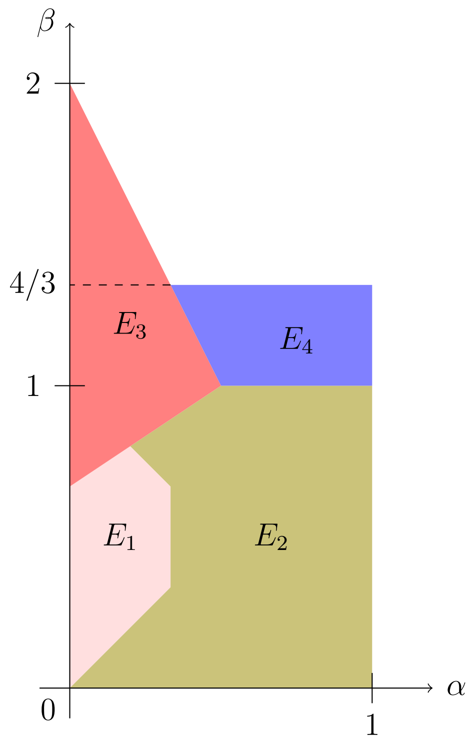

FIGURE 1.1. Ranges where the bounds from Theorem 1.3 are relevant. Here $y = x^\alpha$ and $q = x^\beta$.

We remark that ν in Theorem 1.3 below can also be chosen negative, granted we restrict the sum to integers coprime with q, that is, to n for which n^ν is well-defined modulo q.

In particular, we get the following bound which summarises the non-trivial range allowed by Theorem 1.3:

Corollary 1.4. *Let $\nu \neq 0$ be a fixed positive integer, and let $\varepsilon > 0$. There exists $\delta > 0$, which depends only on ν and ε and such that for a prime $q \geqslant 1$ and real $x \geqslant y \geqslant 2$*

$$S_{\nu,a,q}(x, y) \ll x^{1-\delta}$$

holds uniformly in the range

$$x^{\varepsilon} \leqslant q \leqslant \max\left\{x^{4/3-\varepsilon}, x^{2-\varepsilon}y^{-2}\right\}.$$

These estimates improve the range $x \geqslant q^{1+\varepsilon}y$, accessible via the approach of [13]. We also improve the dependency in y in the estimate

$$|S_{\nu,a,q}(x, y)| \leqslant xq^{-1/4+o(1)} + q^{1/8}x^{3/4+o(1)}y^{1/2}$$

implied by a result of Brüdern and Wooley [8, Theorem 1.1]; compare with (1.4c). We also note that our argument applies to more general sums twisted by multiplicative functions, see Section 5.

2. Bounds on multilinear exponential sums

2.1. **Preliminaries.** We recall that the notations $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leqslant cV$ for some positive constant c, which throughout this work is absolute, unless indicated otherwise. Furthermore, we use $U \asymp V$ in the case when $U \ll V \ll U$.

We also write $U = V^{o(1)}$ if for any fixed ε we have $V^{-\varepsilon} \leqslant |U| \leqslant V^{\varepsilon}$ provided that V is large enough.

For an integer $\ell \neq 0$ we denote by $\tau(\ell)$ the number of positive integer divisors of ℓ, for which we very often use the well-known bound

$$\tau(\ell) = |\ell|^{o(1)} \tag{2.1}$$

as $|\ell| \to \infty$, see [21, Equation (1.81)].

The letter p, with or without subscripts, always denotes a prime number.

Finally, to simplify the statements of our results, we use the notation

$$U \lesssim V$$

to denote that $|U| \leqslant Vx^{o(1)}$ as $x \to \infty$.

2.2. **Bounds arising from the theory of trace function.** We recall two bounds arising from the theory of trace functions. We recall the definition of a non-exceptional trace function $K:\mathbb{Z}\to\mathbb{C}$ in [14, Definition 1.3]. In particular, for any prime q and $\gcd(a,q)=1$ maps of the form $n\mapsto \mathbf{e}_q(an^\nu)$ are trace functions, which are non-exceptional if $\nu\notin\{0,1\}$. See [14, Remark 1.4] for this and other concrete examples of trace functions. The precise definition of exceptional trace functions is given after [14, Remark 1.4], and in particular excludes the case $\nu=1$.

The first result is [14, Theorem 1.17], and concerns Type II sums for trace functions.

Lemma 2.1. *Let K be a non-exceptional trace function. Let q be a prime, and $(\alpha_m),(\beta_n)$ be bounded sequences supported on integers coprime with q in the intervals $[M,2M]$ and $[N,2N]$ respectively, where $M,N\geqslant 1$ and $MN\asymp x$. Then*

$$\sum_{M\leqslant m\leqslant 2M}\sum_{N\leqslant n\leqslant 2N}\alpha_m\beta_n K(mn)\lesssim x\left(q^{-1/4}+M^{-1/2}+q^{1/4}N^{-1/2}\right).$$

The second result concerns special convolution with primes for trace functions. Note that in the case $y>x/2$ is [14, Theorem 1.15, Equation (1.3)] (the m-sum below is reduced to just one term with $m=1$).

Lemma 2.2. *Let q be a prime, and assume $1\leqslant y\leqslant x$. For any $\varepsilon>0$, there exists $\delta>0$ such that*

$$\sum_{y<p\leqslant x}\sum_{m\leqslant x/p}K(mp)\lesssim xq^{-\delta/4}+x^{1-\delta}q^{(3/4+\varepsilon)\delta}, \tag{2.2a}$$

$$\sum_{y<p_1\leqslant p_2\leqslant x}\sum_{m\leqslant x/p_1p_2}K(mp_1p_2)\lesssim xq^{-\delta/4}+x^{1-\delta}q^{(3/4+\varepsilon)\delta}. \tag{2.2b}$$

Proof. Consider first (2.2a). Let $\Delta\in(1/x,1]$ be some parameter to be fixed later. Using a partition of unity as in [14, p. 1717], we have an upper-bound

$$\sum_{y<p\leqslant x}\sum_{m\leqslant x/p}K(mp)\lesssim x\Delta+\left|\sum_{p}\sum_{m}V_0(mp)V_1(p)K(mp)\right| \tag{2.3}$$

for some functions V_0,V_1 which are smooth on $\mathbb{R}$, with supports

$$\operatorname{supp}V_0\subseteq[L,2L],\qquad \operatorname{supp}V_1\subseteq[P,2P],$$

for some L and P satisfying $1\leqslant L\ll x$ and $y\ll P\ll x$, and moreover V_0,V_1 have derivatives bounded by [14, Equation (1.1)], that is,

$$x^kV_h^{(k)}(x)\ll Q^k,\qquad h=0,1, \tag{2.4}$$

for $k = 0, 1, \ldots$, with $Q = \Delta^{-1}$ (all the implied constant throughout the proof may depend k).

In particular, on the right hand side of (2.3) and in several formulas below, the sums over p and m are both supported over finite sets.

Let $\widehat{V}_h$ be the Mellin inverse of V_h. For any fixed $k \geqslant 0$ we have

$$\widehat{V}_h(s) \ll \left(\frac{Q}{1+|s|}\right)^k, \qquad h = 0, 1. \tag{2.5}$$

Hence

$$V_h(y) = \frac{1}{2\pi i} \int_{|t| \leqslant \Delta^{-2}} \widehat{V}(it) y^{-it} dt + O(\Delta), \qquad h = 0, 1.$$

Thus, using above representation for V_1 and the bound

$$\int_{|t| \leqslant \Delta^{-2}} \left|\widehat{V}_h(it)\right| dt \ll Q \log Q \lesssim \Delta^{-1}$$

which follows from (2.5) taken with $k = 1$ (and our assumption $\Delta > 1/x$) we derive

$$\begin{aligned} &\left|\sum_p \sum_m V_0(mp) V_1(p) K(mp)\right| \\ &\qquad \lesssim x\Delta + \Delta^{-1} \sup_{|t| \leqslant \Delta^{-2}} \left|\sum_p \sum_m p^{it} V_0(mp) K(mp)\right|. \end{aligned} \tag{2.6}$$

Next, we fix t with

$$|t| \leqslant \Delta^{-2} \tag{2.7}$$

and use the Heath-Brown identity for primes [18] followed by a partition of unity, as in [14, Section 4.1]. It is also convenient to rename the variable m in (2.6) as n_1. Thus, after fixing an arbitrary integer $J \geq 1$ and setting $J^* = J + 1$, we are reduced to bound sums Σ of the shape

$$\begin{aligned} \Sigma(\mathbf{M}, \mathbf{N}) = &\sum_{m_1, \ldots, m_J} \cdots \sum \alpha_1(m_1) \cdots \alpha_J(m_J) \\ &\times \sum_{n_1, \ldots, n_{J^*}} \cdots \sum V_1(n_1) V_2(n_2) \cdots V_{J^*}(n_{J^*}) \\ &\qquad \times (n_2 \ldots n_{J^*})^{it} V_0(m_1 \cdots m_J n_1 \cdots n_{J^*}), \end{aligned} \tag{2.8}$$

where

- $\mathbf{M} = (M_1, \ldots, M_J)$ and $\mathbf{N} = (N_1, \ldots, N_{J^*})$ are tuples of parameters in $[1/2, 2x]^{2J}$ and $[1/2, 2x]^{2J^*}$, respectively, which satisfy
$$N_2 \geqslant \ldots \geqslant N_{J^*}, \qquad M_1, \ldots, M_J \leqslant x^{1/J},$$
$$M_1 \cdots M_J N_1 \cdots N_{J^*} \ll X$$
with an implied constant depending only of J;
- the weights $\alpha_j(m_j)$ are bounded and supported in $[M_j/2, M_j]$, $j = 1, \ldots, J$;
- the maps V_j are compactly supported in $[N_j/2, N_j]$ and their derivatives satisfy
$$y^k V_j^{(k)}(y) \ll 1, \qquad j = 1, \ldots, J^*,$$
for every integer $k \geq 0$.

For each $j = 2, \ldots, J^*$, let $\widetilde{V}_j(y) = V_j(y) y^{it}$, which, in view of (2.7), satisfies

$$y^k \widetilde{V}_j^{(k)}(y) \ll \Delta^{-2k} \tag{2.9}$$

for each fixed integer $k \geq 0$. Hence we can rewrite Σ as

$$\begin{aligned}\Sigma(\mathbf{M}, \mathbf{N}) = \sum_{m_1, \ldots, m_J} \cdots \sum \alpha_1(m_1) \cdots \alpha_J(m_J) \\ \times \sum_{n_1, \ldots, n_{J^*}} \cdots \sum V_1(n_1) \widetilde{V}_2(n_2) \cdots \widetilde{V}_{J^*}(n_{J^*}) \\ \times V_0(m_1 \cdots m_J n_1 \cdots n_{J^*}).\end{aligned}$$

Compared with [14, Equation (4.1)], the only difference is the growth of the derivatives of $\widetilde{V}_2, \ldots, \widetilde{V}_{J^*}$. From here, the arguments in [14, Section 4.2], which essentially use only cancellations with respect to two variables of summation n_1 and n_2. Recalling (2.4) and (2.9), we see that we have $Q_U, Q_V, Q_W \leqslant Q^{O(1)}$ in the condition of [14, Theorem 1.16], and thus the bound [14, Equation (4.2)] becomes

$$\Sigma(\mathbf{M}, \mathbf{N}) \ll Q^A x \left(1 + \frac{q}{N_1 N_2}\right)^{1/2} q^{-1/8+\varepsilon}$$

for some A, depending only on ε. Following these arguments, we obtain

$$\sum_{y < p \leqslant x} \sum_{m \leqslant x/p} K(mp) \lesssim x\Delta + q^{\varepsilon} x \Delta^{-A} (1 + q/x)^{1/6} q^{-1/24},$$

from which the desired result follows upon balancing Δ (which, as one can easily see, satisfies the condition $\Delta > 1/x$).

The bound (2.2b) follows by an identical argument. $\square$

2.3. **Bounds with monomials: bilinear forms and primes.** The following estimate is Vinogradov's classical result for bilinear exponential sums when $\nu = 1$, extended to $\nu \neq 0$ through the use of Dirichlet characters.

Lemma 2.3. *Let $\nu \neq 0$ be an integer. Let $M, N, x \geqslant 2$, and $(\alpha_m), (\beta_n)$ with $|\alpha_m| \leqslant 1$ and $|\beta_n| \leqslant 1$ be sequences supported on integers coprime with q in dyadic intervals $[M, 2M]$, $[N, 2N]$ respectively. Then*

$$\sum_{\substack{mn \leqslant x \\ M \leqslant m \leqslant 2M \\ N \leqslant n \leqslant 2N}} \alpha_m \beta_n \, \mathbf{e}_q(a(mn)^\nu) \lesssim MN \left(M^{-1/2} + N^{-1/2} + q^{-1/2} + (MN/q)^{-1/2} \right),$$

where the coprimality assumption $\gcd(mn, q) = 1$ on the supports of (α_m) and (β_n) can be removed if $\nu \geqslant 1$.

Proof. Clearly, we can assume that $MN \leqslant x$ as as otherwise the sum is void. Hence all terms of the shape $(MN)^{o(1)}$ can be absorbed in $\lesssim$.

We separate analytically the variables m, n in the condition $mn \leqslant x$ by means of [21, Lemma 13.11], getting

$$\sum_{\substack{mn \leqslant x \\ M \leqslant m \leqslant 2M \\ N \leqslant n \leqslant 2N}} \alpha_m \beta_n \, \mathbf{e}_q(a(mn)^\nu) \lesssim \sup_{t \in \mathbb{R}} \left| \sum_{\substack{M \leqslant m \leqslant 2M \\ N \leqslant n \leqslant 2N}} \alpha_{m,t} \beta_{n,t} \, \mathbf{e}_q(a(mn)^\nu) \right|, \tag{2.10}$$

where $\alpha_{m,t} = \alpha_m m^{it}$ and similarly for $\beta_{n,t}$. We now partition the last sum into at most $(M/q+1)(N/q+1) \ll q^{-2} \max\{M, q\} \max\{N, q\}$ sums with ranges of variables m and n of length $X = \min\{M, q\}$ and $Y = \min\{N, q\}$, respectively. By a classical result, see, for example [32, Chapter VI, Exercise 14.a], each of these sums is bounded by $\sqrt{qXY}$. Since $\max\{A, B\} \min\{A, B\} = AB$ this leads to the bound

$$\begin{aligned} q^{-2} \max\{M, q\} \max\{N, q\} & \sqrt{q \min\{M, q\} \min\{N.q\}} \\ &= q^{-2} \sqrt{\max\{M, q\} \max\{N, q\} q^3 MN} \\ &\leqslant (MN)^{1/2} q^{-1/2} (M+q)^{1/2} (N+q)^{1/2}, \end{aligned}$$

which after substitution in (2.10) concludes the proof. □

The following result is a variant of the classical bound for exponential sums over primes, with an additional convolution.

Lemma 2.4. *For any $2 \leqslant y \leqslant x$, we have*

$$\sum_{y<p\leqslant x} \sum_{m\leqslant x/p} \mathbf{e}_q(amp) \lesssim x^{4/5} + x^{1/2}(x/q+q)^{1/2}.$$

Proof. We split the sum over p as

$$\sum_{y<p\leqslant x} \sum_{m\leqslant x/p} \mathbf{e}_q(amp) = S_1 + S_2$$

where S_1 is subject to $p \leqslant x^{4/5}$ and S_2 is the complementary sum. To S_1 we apply [21, Equation (13.46)] with the choice $M = x^{4/5}$, getting the admissible bound

$$S_1 \lesssim x^{4/5} + x/q + q.$$

To evaluate S_2, by partial summation, a trivial bound on the contribution of prime powers, and splitting in dyadic intervals, we get

$$S_2 \ll x^{3/5} + \sup_{M\leqslant x^{1/5}} (S_{21}(M) + S_{22}(M)),$$

where

$$S_{21}(M) = \frac{1}{\log x} \sum_{M/2<m\leqslant M} \sum_{\substack{y<n\leqslant x/m \\ n>x^{5/6}}} \Lambda(n)\, \mathbf{e}_q(amn),$$

$$S_{22}(M) = \int_2^x \frac{1}{t(\log t)^2} \sum_{M/2<m\leqslant M} \sum_{\substack{y<n\leqslant x/m \\ x^{5/6}<n<t}} \Lambda(m)\, \mathbf{e}_q(amn) dt.$$

Let us focus on $S_{21}(M)$. We use the Vaughan identity, see [21, Equation (13.39)] with parameters y, z there replaced by $x^{2/5}/M$ and $x^{2/5}$ respectively, getting

$$S_{21} \log x \ll |\Sigma_1| + |\Sigma_2| + |\Sigma_3|$$

where

$$\Sigma_1 = \sum_{M/2<m\leqslant M} \sum_{b\leqslant x^{2/5}/M} \sum_{n:\ bn\in\mathcal{I}_m} \mu(b)(\log n)\, \mathbf{e}_q(ambn),$$

$$\Sigma_2 = \sum_{M/2<m\leqslant M} \sum_{b\leqslant x^{2/5}/M} \sum_{c\leqslant x^{2/5}} \sum_{n:\ bcn\in\mathcal{I}_m} \mu(b)\Lambda(c)\, \mathbf{e}_q(ambcn),$$

$$\Sigma_3 = \sum_{M/2<m\leqslant M} \sum_{b> x^{2/5}/M} \sum_{c> x^{2/5}} \sum_{n:\ bcn\in\mathcal{I}_m} \mu(b)\Lambda(c)\, \mathbf{e}_q(ambcn),$$

and $\mathcal{I}_m = \mathbb{Z} \cap (\max\{y, x^{5/6}\}, x/m]$.

Then we follow the steps in the proof of [21, Theorem 13.6]. In particular, we note that by [21, Equation (8.6)], for any $c \in \mathbb{Z}$ and $z > 0$, we have

$$\sum_{1 \leqslant n \leqslant z} \mathbf{e}_q(cn) \ll \max\left\{z, \|c/q\|^{-1}\right\},$$

where $\|\xi\| = \min\{|\xi - k| :\ k \in \mathbb{Z}\}$. Now, using partial summation, we see that the first sum Σ_1 is bounded by

$$\begin{aligned} \Sigma_1 &\ll \int_1^x \sum_{M/2 < m \leqslant M} \sum_{b \leqslant x^{2/5}/M} \left| \sum_{n:\ bn \in I_m, n > t} \mathbf{e}_q(ambn) \right| \frac{dt}{t} \\ &\ll \int_1^x \sum_{M/2 < m \leqslant M} \sum_{b \leqslant x^{2/5}/M} \max\left\{\frac{x}{bm}, \|amb/q\|^{-1}\right\} \frac{dt}{t} \\ &\ll \sum_{\ell \leq x^{2/5}} \tau(\ell) \max\left\{\frac{x}{\ell}, \|a\ell/q\|^{-1}\right\}. \end{aligned} \tag{2.11}$$

Recalling the bound (2.1), we derive

$$\Sigma_1 \lesssim x^{2/5} + x/q + q.$$

The second sum Σ_2 is bounded similarly, with the upper bound on ℓ being $\ell \leq x^{4/5}$, and thus we have

$$\Sigma_2 \lesssim x^{4/5} + x/q + q.$$

Finally the third sum Σ_2 is bounded using Lemma 2.3 by

$$\Sigma_3 \lesssim x\left((x^{-2/5})^{1/2} + q^{-1/2} + (x/q)^{-1/2}\right) \ll x^{4/5} + \frac{x}{q^{1/2}} + (qx)^{1/2},$$

which dominates the above bounds on Σ_1 and Σ_2. Combining these estimates, we deduce

$$S_{21}(M) \lesssim x^{4/5} + \frac{x}{q^{1/2}} + (qx)^{1/2}.$$

The same upper bound on $S_{22}(M)$, and therefore on S_2 follows by an identical analysis. $\square$

For convolution with one prime for non-linear phases we get the following slightly worse estimate.

Lemma 2.5. *Let $\nu \in \mathbb{Z} \setminus \{0, 1\}$. For any $2 \leqslant y \leqslant x$, we have*

$$\sum_{y < p \leqslant x} \sum_{m \leqslant x/p} \mathbf{e}_q(a(mp)^\nu) \lesssim xy^{-1/2} + x^{3/4}q^{1/8} + x^{1/2}(x/q + q)^{1/2}.$$

Proof. Let S_ν be the sum on the left-hand side, and write

$$S_\nu \ll \sup_{M \leqslant x/y} S_\nu(M),$$

where $S_\nu(M)$ is the contribution of those $m \in (M/2, M)$. The contribution of $M > x^{1/2}q^{-1/4}$ is dealt with using Lemma 2.3, which gives

$$S_\nu(M) \lesssim xM^{-1/2} + (xM)^{1/2} + xq^{-1/2} + (xq)^{1/2}.$$

Taking the supremum over M satisfying $x^{1/2}q^{-1/4} < M \leqslant x/y$ gives an acceptable upper bound.

To deal with the contribution of those $M \leqslant x^{1/2}q^{-1/4}$ we follow the arguments in [30], with the choice of parameters $v = q$ and $u = x^{1/2}q^{-1/4}M^{-1}$, in a way analogous to the proof of Lemma 2.4. □

Finally we also require the following estimate for double-sums with convolution with two primes for monomial phases.

Lemma 2.6. *Let $\nu \neq 0$ and $j \geqslant 2$ be integers. For any $2 \leqslant y \leqslant x$, we have*

$$\sum_{y<p_1<\cdots<p_j} \sum_{m \leqslant x/p_1\cdots p_j} \mathbf{e}_q(a(mp_1\cdots p_j)^\nu)$$
$$\lesssim x^{1/2}(x/q+q)^{1/2} + \begin{cases} xy^{-1/2} & \text{if } \nu \neq 1, \\ \min\{xy^{-1/2}, x^{4/5}\} & \text{if } \nu = 1. \end{cases}$$

Proof. First assume that either $\nu \neq 1$ or $y \geqslant x^{2/5}$. Note that terms with $p_k = p_\ell$ for some $k \neq \ell$ can be included in the sum at a cost $O(x/\sqrt{y})$. We may therefore, by symmetry, relax the conditions $p_k < p_{k+1}$. Next, we group the variables (p_1, m) and $(p_2, \ldots, p_j)$ together and let

$$\beta_\ell = \sum_{\substack{p_1 \mid \ell \\ p_1 > y}} 1, \qquad \gamma_n = \sum_{\substack{p_2,\ldots,p_k \mid n \\ p_i > y}} 1,$$

so that our sum can be replaced with

$$S = \sum_{y<n\leqslant x} \sum_{y<\ell\leqslant x/n} \beta_\ell \gamma_n \, \mathbf{e}_q(a(\ell n)^\nu).$$

Note that $|\beta_\ell| \leqslant \omega(\ell) \lesssim 1$, and similarly $|\gamma_n| \leqslant \omega(n)^{k-1} \lesssim 1$, where $\omega(\ell)$ is the number of distinct prime divisors of ℓ. By Lemma 2.3, we deduce

$$S \lesssim x(y^{-1/2} + q^{-1/2} + (x/q)^{-1/2}), \tag{2.12}$$

which gives the claimed estimate for general ν.

Suppose next that $\nu = 1$ and $y < x^{2/5}$. We use j times the Heath-Brown identity for primes, as we have done earlier in (2.8), which brings us to bound a finite number of sums of the shape

$$T = \sum_{\substack{m_1,\dots,m_J \\ m_i \leqslant x^{1/5} \\ (m_1,\dots,m_J,n_1,\dots,n_{J^*}) \in \mathcal{U}}} \sum_{n_1,\dots,n_{J^*}} \mu(m_1)\cdots\mu(m_J) \times V_1(n_1)\cdots V_{J^*}(n_{J^*})\, \mathbf{e}_q(am_1\cdots m_J n_1 \cdots n_{J^*}), \tag{2.13}$$

where $\mathcal{U} \subseteq \mathbb{R}^{J+J^*}$ accounts for the various inequalities that involve p_i, and $V_i(t)$ is either 1 or $\log t$. Concerning $\mathcal{U}$, we keep only the information that

$$\{(\log m_1, \dots, \log n_{J^*}) :\ (m_1, \dots, n_{J^*}) \in \mathcal{U}\} \tag{2.14}$$

is a convex set, and in fact an intersection of half-spaces.

We partition the sum (2.13) in dyadic intervals $m_i \in [M_i, 2M_i]$ and $n_i \in [N_i, 2N_i]$, with $M_i \leqslant x^{1/5}$. If there is an index i such that $N_i \geqslant x^{1/5}$, we sum over n_i first (which we rename into n) to get a sum of the shape

$$T \lesssim \sum_{\substack{\ell_1,\dots,\ell_L \\ \ell_1\cdots\ell_L \ll x^{4/5}}} \left| \sum_{n \in I_{\ell_1,\dots,\ell_L}} V_i(n)\, \mathbf{e}_q(a\ell_1\cdots\ell_L n) \right|,$$

where $L = J + J^* - 1 = 2J$, the set $I_{\ell_1,\dots,\ell_L}$ is an interval by convexity of (2.14), and it is contained in $[1, x/(\ell_1\cdots\ell_L)]$. Therefore, using a Type I estimate [21, Equation (13.46)] as we have done earlier in (2.11), gives a bound

$$T \lesssim x^{4/5} + xq^{-1} + q. \tag{2.15}$$

If $N_i < x^{1/5}$ for all i, then since $M_i \leqslant x^{1/5}$ as well, we may group variables in such a way as to obtain a Type II of the shape [21, Equation (13.48)] with $x^{2/5} \ll M \ll x^{3/5}$, and then we get a bound

$$T \lesssim x^{4/5} + xq^{-1/2} + (xq)^{1/2}. \tag{2.16}$$

We observe that the desired bound is trivial for $q \geqslant x$. Otherwise $q \leqslant (xq)^{1/2}$, and taking the weakest of the bounds (2.15) and (2.16), also also recalling that (2.12) also holds for $\nu = 1$, we conclude the proof. $\square$

2.4. **Combinatorial decomposition of integers without large prime factors.** The previous results are used in conjunction with the following two combinatorial decomposition for the indicator function of smooth numbers, which are relevant for small y and large y respectively.

Lemma 2.7. *For $2 \leqslant y \leqslant x$ and any bounded map $f : \mathbb{N} \to \mathbb{C}$, for any positive $w \leqslant x$, for some sequences (α_m), (β_n) of bounded L_∞-norm, we have*

$$\sum_{n \in \mathcal{S}(x,y)} f(n) \lesssim w + \sup_{\substack{w \leqslant M \leqslant wy \\ MN \leqslant x}} \left| \sum_{m \in [M,2M]} \sum_{n \in [N,2N]} \alpha_m \beta_n f(mn) \right|,$$

where the supremum is over all $M, N \geqslant 1$ with the indicated conditions.

Proof. The argument is based on the classical combinatorial partition of y-smooth integers, see, for example [24, p. 1369] or [31, Lemma 10.1]. Then we proceed as in [12, Lemma 3.4]. First we bound trivially the contribution of $n \leqslant w$. Then we factor each $n \in \mathcal{S}(x,y)$ uniquely as $n = km$ with $w \leqslant k < wP(k)$ and $P(k) \leqslant p(m)$, where $p(m)$ is the smallest prime divisor of m. Finally we separate multiplicatively k and m analytically using [21, Lemma 13.11]. More precisely, exactly as in [13, Section 9], we write

$$\sum_{\substack{n \in \mathcal{S}(x,y) \\ n \geqslant w}} f(n) = \sum_{\substack{w < k \leqslant wP(k) \\ P(k) \leqslant y}} \sum_{\substack{m \in \mathcal{S}(x/k,y) \\ p(m) \geq P(k)}} f(km).$$

The condition $p(m) \geq P(k)$ involves integers on both sides of this inequality of size at most y. We detect this condition by means of [21, Lemma 13.11], getting

$$\sum_{\substack{n \in \mathcal{S}(x,y) \\ n \geqslant w}} f(n) \lesssim \sup_{t \in \mathbb{R}} |\mathsf{S}(t)|,$$

where

$$\mathsf{S}(t) = \sum_{\substack{w < k \leqslant wP(k) \\ P(k) \leqslant y}} P(k)^{it} \sum_{m \in \mathcal{S}(x/k,y)} p(m)^{-it} f(km).$$

We split the sum $\mathsf{S}(t)$ into sums $\mathsf{S}(t,K,M)$ over dyadic intervals $K \leqslant k < 2K$, $M \leqslant m < 2M$ with $w \leqslant K \leqslant wy$ and $KM \leqslant x$, and we write accordingly

$$\sum_{\substack{n \in \mathcal{S}(x,y) \\ n \geqslant w}} f(n) \lesssim \sup_{\substack{w \leqslant K \leqslant wy \\ KM \leqslant x}} |\mathsf{S}(t,K,M)|.$$

Renaming the variables, we derive the desired result. □

We also need yet another simple combinatorial identity.

Lemma 2.8. *Let $f:\mathbb{N}\to\mathbb{C}$ be a bounded map and let r be a positive integer. For $x^{1/(r+1)} < y \leqslant x^{1/r}$ we have*

$$\sum_{n\in\mathcal{S}(x,y)} f(n) = \sum_{n\leqslant x} f(n) + \sum_{j=1}^{r}(-1)^j \sum_{y<p_1\leqslant\cdots\leqslant p_j}\ \sum_{m\leqslant x/p_1\cdots p_j} f(mp_1\cdots p_j).$$

Proof. This is the classical Buchstab identity [9], see, for example, [29, Theorem III.4]. □

3. Proof of Theorem 1.1

As in the proof of [13, Theorem 13], we choose some parameter w to be chosen later subject to $1\leqslant w\leqslant x$ (we however do not set $w=x^{1/2}$ as in [13]). We apply Lemma 2.7 and use Lemma 2.3 to bound the resulting sums. The four terms in the bound of Lemma 2.3 can be estimated as

$$\begin{aligned}
&M^{1/2}N \leqslant M^{1/2}(x/M)\leqslant xw^{-1/2},\\
&MN^{1/2}\leqslant M^{1/2}x^{1/2}\leqslant (wxy)^{1/2},\\
&MNq^{-1/2}\leqslant xq^{-1/2},\\
&MN(MN/q)^{-1/2} = (MNq)^{1/2}\leqslant (xq)^{1/2}.
\end{aligned}$$

Hence we obtain

$$S_{a,q}(x,y)\lesssim w + x\left(w^{-1/2} + (x/(wy))^{-1/2} + q^{-1/2} + (x/q)^{-1/2}\right).$$

We now pick $w=(x/y)^{1/2}$ which indeed satisfies $1\leqslant w\leqslant x$, and get the bound

$$S_{a,q}(x,y)\lesssim x\left((x/y)^{-1/4} + q^{-1/2} + (x/q)^{-1/2}\right). \tag{3.1}$$

This proves Theorem 1.1 when $y\leqslant x^{1/5}$, as then $(x/y)^{-1/4}\leqslant x^{-1/5}$.

We now focus on the range $x^{1/5}<y\leqslant x$. By Lemma 2.8 we have

$$S_{a,q}(x,y) = \sum_{n\leqslant x}\mathbf{e}_q(an) + \sum_{j=1}^{5}(-1)^j\sum_{y<p_1\leqslant\cdots\leqslant p_j}\ \sum_{m\leqslant x/p_1\cdots p_j}\mathbf{e}_q(amp_1\cdots p_j).$$

The first sum is trivially $O(q)$, which is admissible since $q\leqslant (xq)^{1/2}$ for $x\geqslant q$, which we can always assume. The last five sums are bounded, using Lemmas 2.4 (for $j=1$) and 2.6 (for $2\leqslant j\leqslant 5$), by

$$\begin{aligned}
&\sum_{y<p_1<\cdots<p_j}\ \sum_{m\leqslant x/p_1\cdots p_j}\mathbf{e}_q(amp_1\cdots p_j)\\
&\qquad\qquad \lesssim x\left(x^{-1/5} + q^{-1/2} + (x/q)^{-1/2}\right).
\end{aligned}$$

This proves Theorem 1.1 when $x^{1/5} < y \leqslant x$.

4. Proof of Theorem 1.3

4.1. **Preliminary splitting.** We now assume that q is prime. Removing the contribution of those integers divisible by q, we get

$$S_{\nu,a,q}(x,y) = \sum_{\substack{n\in\mathcal{S}(x,y)\\ \gcd(n,q)=1}} \mathbf{e}_q(an^\nu) + O(x/q).$$

4.2. **Proof of (1.4a).** The proof of (1.4a) is identical to the proof of the bound (3.1), since Lemma 2.3 holds for any ν (and actually, for any non-exceptional trace function).

4.3. **Proof of (1.4b).** We proceed as in the proof of Theorem 1.1, except that we use Lemma 2.5 (instead of Lemma 2.4). The details are identical.

4.4. **Proof of (1.4c).** Let $1 \leqslant w \leqslant x$ be a parameter. Using Lemma 2.7, followed by Lemma 2.1, we get

$$\begin{aligned} S_{\nu,a,q}(x,y) \lesssim xq^{-1/4} &+ w \\ &+ x \sup_{w\leqslant M\leqslant wy} \min\bigl\{M^{-1/2} + x^{-1/2}q^{1/4}M^{1/2}, \\ &\qquad\qquad x^{-1/2}M^{1/2} + q^{1/4}M^{-1/2}\bigr\}, \end{aligned}$$

where we used the symmetry of the bounds of Lemma 2.1 with respect to $M \leftrightarrow N$.

Write $w = x^\omega$, $y = x^\alpha$, $q = x^\beta$, $M = x^\mu$, and let

$$\eta(\mu) = \begin{cases} \min\{\mu/2, 1/2 - \beta/4 - \mu/2\}, & (0 \leqslant \mu \leqslant 1/2), \\ \min\{\mu/2 - \beta/4, 1/2 - \mu/2\}, & (1/2 < \mu \leqslant 1). \end{cases}$$

Recalling $w \leqslant M \leqslant wy$ we see that only the range $\omega \leqslant \mu \leqslant \omega + \alpha$ is relevant to us. That is, we are interested in choosing ω which maximises

$$\kappa = \min_{\omega\leqslant\mu\leqslant\omega+\alpha} \eta(\mu).$$

Note that we dropped the condition $\mu \leqslant 1$ as for $\mu \geqslant 1$ we have $\eta(\mu) < 0$ and the result is trivial.

The bound above reads

$$S_{\nu,a,q}(x,y) \lesssim x^{1-\beta/4} + x^\omega + x^{1-\kappa} \tag{4.1}$$

For $q \leqslant x$, which means $\beta \leqslant 1$, we maximise κ by setting

$$\omega = \begin{cases} 1/2 - \beta/4 - \alpha/2, & (\alpha < \beta/2), \\ (1-\beta)/2, & (\beta/2 \leqslant \alpha < \beta), \\ (1-\alpha)/2, & (\beta \leqslant \alpha \leqslant 1), \end{cases}$$

which gives in all cases $\kappa = \omega/2$.

Indeed, our optimisation problem has a natural interpretation of fitting a horizontal interval $\mathcal{I}$ of length α at the maximal height under the plot of the function $\eta(\mu)$ which looks like a union of two symmetric peaks, see Figure 4.1. There are there different regimes which correspond to the above choice of ω:

- $\mathcal{I}$ fits entirely inside of one peak, see the solid line on Figure 4.1;
- $\mathcal{I}$ fits just under the intersection point of the peaks, see the dashed line on Figure 4.1;
- $\mathcal{I}$ can only be fit strictly below the intersection point of the peaks and stretches from one edge of the plot to another, see the dotted line on Figure 4.1;

In fact, it is easy to see that in the first case there is yet another optimal choice of ω, which in the second case case we have infinitely many possibilities. Howeve, since the bound (4.1) contains the term x^{ω} we always select the smallest admissible value.

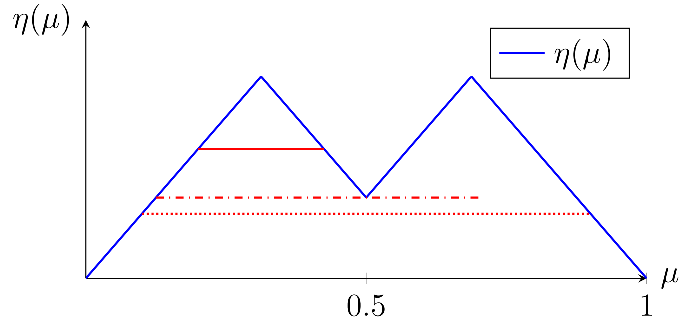

FIGURE 4.1. Three different regimes of α and β.

Noting that $\omega \leqslant 1/2 \leqslant 1 - \beta/4$ in all cases, we get for $q \leqslant x$ the bound

$$S_{\nu,a,q}(x,y) \lesssim xq^{-1/4} + x \times \begin{cases} x^{-1/4}q^{1/8}y^{1/4}, & (1 \leqslant y \leqslant q^{1/2}), \\ (x/q)^{-1/4}, & (q^{1/2} < y \leqslant q), \\ (x/y)^{-1/4}, & (q < y \leqslant x). \end{cases}$$

It is easily checked that this coincides with (1.4c).

For $x < q \leqslant x^2$, we assume that $y < xq^{-1/2}$, for otherwise the claimed bound (1.4c) is trivial. This translates to $\alpha < 1 - \beta/2$. We optimise the bound (4.1) by setting

$$\omega = 1/2 - \beta/4 - \alpha/2,$$

and we obtain

$$S_{\nu,a,q}(x,y) \lesssim xq^{-1/4} + x(x/y)^{-1/4}q^{1/8}$$

in accordance with (1.4c).

4.5. **Proof of (1.4d).** First we note that for $y \leqslant x^{1/3}$, the bound (1.4c) which we have just proven implies

$$S_{\nu,a,q}(x,y) \lesssim x(q^{-1/2} + x^{-4/3}q)^{1/8},$$

which implies (1.4d), reducing the value of δ if necessary. We may thus assume $x^{1/3} < y \leqslant x$.

Assume first that $x^{1/2} < y \leqslant x$. We use Lemma 2.8 and get

$$S_{\nu,a,q}(x,y) = O(x/q) + \sum_{\substack{n \leqslant x \\ \gcd(n,q)=1}} \mathbf{e}_q(an^\nu) - \sum_{y < p \leqslant x} \sum_{m \leqslant x/p} \mathbf{e}_q(a(mp)^\nu).$$

Using the Weil bound [34], coupled with the completing technique [21, Section 12.2], and periodicity, the first sum on the right-hand side is bounded by

$$\sum_{\substack{n \leqslant x \\ \gcd(n,q)=1}} \mathbf{e}_q(an^\nu) \lesssim q^{1/2} + xq^{-1/2}.$$

To bound the second sum, we appeal to the bound (2.2a) of Lemma 2.2. It follows that for each $\varepsilon > 0$, there exists $\delta > 0$ for which we have

$$S_{\nu,a,q}(x,y) \lesssim q^{1/2} + xq^{-1/2} + q^{-\delta/4} + q^{(3/4+\varepsilon)\delta}x^{-\delta}.$$

Reducing δ if necessary, the first two terms are absorbed by the last two terms, and we obtain (1.4d) for $y > x^{1/2}$.

The case $x^{1/3} < y \leqslant x^{1/2}$ similar: upon using Lemma 2.8, we are to bound an additional sum with $j = 2$, namely

$$\sum_{y < p_1 \leqslant p_2 \leqslant x} \sum_{m \leqslant x/p_1p_2} \mathbf{e}_q(a(mp_1p_2)^\nu),$$

for which an admissible bound is provided by (2.2b) of Lemma 2.2 .

5. Comments

We have already mentioned that the bound of Theorem 1.1 is nontrivial in essentially optimal range $x \geqslant q^{1+\varepsilon}$ with an arbitrary fixed $\varepsilon > 0$. However, the range where Theorem 1.3 gives a power saving is unlikely to be the best possible. In fact, for $\nu \geqslant 1$ one can expect nontrivial bounds starting already from $x \geqslant q^{1/\nu+\varepsilon}$ with an arbitrary fixed $\varepsilon > 0$. One of the possibilities to extend this range is via the use of some other bounds on the bilinear sum which appears on the right hand side of (2.10), exploiting the structure of the argument, instead of the generic bound from [32, Chapter VI, Exercise 14.a]. For example, a double application of the Hölder inequality leads to the following inequality, which in several modifications has appeared in a large number of works and follows the steps in the proof of [22, Theorem 3]. Namely, for any integer $k, \ell \geqslant 1$, we have

$$\left|\sum_{\substack{M\leqslant m\leqslant 2M\\ N\leqslant n\leqslant 2N}} \alpha_m \beta_n \, \mathbf{e}_q(a(mn)^\nu)\right|^{2k\ell} \leqslant qM^{1-1/\ell}N^{1-1/k}\left(T_k(M)T_\ell(N)\right)^{1/k\ell},$$

where $T_k(M)$ is the number of solutions to the congrunce.

$$m_1^\nu + \ldots + m_k^\nu \equiv m_{k+1}^\nu + \ldots m_{2k}^\nu \pmod q, \quad M \leqslant m_1, \ldots, m_{2k} \leqslant 2M,$$

and similarly for $T_\ell(N)$. Note that with $k = \ell = 1$ this is exactly the bound we have used to in the proof of Lemma 2.3. To estimate $T_k(M)$ and $T_\ell(N)$ for $k, \ell \geqslant 2$ and $\nu \geqslant 2$ one can use, for example, [23, Theorems 1.1 and 1.2]. Furthermore, for $\nu \leqslant -1$, one can also use various bounds of Bourgain and Garaev [1], Heath-Brown [19] and Pierce [27].

Our approach can be adjusted to obtain similar bounds to several variations of the sums $S_{a,q}(x, y)$ and $S_{\nu,a,q}(x, y)$. For example, these includes sums twisted by multiplicative functions such as

$$S_{a,q}(f; x, y) = \sum_{n\in\mathcal{S}(x,y)} f(n)\, \mathbf{e}_q(an),$$

and more generally

$$S_{\nu,a,q}(f; x, y) = \sum_{n\in\mathcal{S}(x,y)} f(n)\, \mathbf{e}_q(an^\nu), \quad \nu = \pm 1, \pm 2, \ldots,$$

with a multiplicative function $f(n)$. Sums $S_{a,q}(f;x,y)$ have also been studied [2, Proposition 1], see also [5, Section 10.2]. If the function f is completely multiplicative, such as a multiplicative character, our argument proceeds without any changes besides small typographical adjustments and so the bounds of Theorems 1.1 and 1.3 also apply to $S_{a,q}(f;x,y)$ and $S_{\nu,a,q}(f;x,y)$ (with an additional factor $\max_{n\leqslant x}|f(n)|$).

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